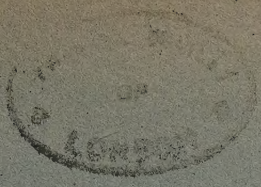

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On the Principle of the Light-Velocity

By

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In the special theory of relativity, we assume that the velocity of the light is absolutely constant for any observer; whence it follows that the velocity of the light is maximum. In the general theory of relativity, the assumption on the light becomes $ds^2=o$. If we intend to treat the theories deductively, we must at first assume that $ds^2=o$ for the light. Thus in the deductive treatment, the principle on the light is not banal for understanding. In this note its meaning is considered.

The statical gravitation-field of Einstein is, after Schwalzschild,[1]

$$ds^2=-\frac{dr^2}{\gamma}-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\gamma dt^2, \qquad (1)$$

$$\gamma=1-\frac{2m}{r}.$$

The motion of a particle in the plane $\varphi=$const., without any *masses*, is given by the equations

$$\left.\begin{aligned} r^2\frac{d\theta}{ds}&=h,\\ \frac{dt}{ds}&=\frac{k}{\gamma},\\ \left(\frac{dr}{ds}\right)^2+r^2\left(\frac{d\theta}{ds}\right)^2&=k^2-1+\frac{2m}{r}-\frac{2mh^2}{r^3}, \end{aligned}\right\} \qquad (2)$$

1 Eddington, Report on the relativity theory of gravitation.

where h and k are integration-constants.

In newtonion dynamics of the planetary motion, we have

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}-\frac{m}{a}, \tag{3}$$

where m is the gravitational mass of the sun, a, the semimajor axis of the orbit. By the principle of equivalence, we have

$$k^2-1=-\frac{m}{a}, \quad \text{or} \quad k=\sqrt{1-\frac{m}{a}}. \tag{4}$$

Therefore we must limit the variation of the constant k. If a be small or m be great k would be imaginary and the principle of equivalence would be fatal. Now if we restrict the principle in the limit that (4) should be true, we have nothing to say. But we *extend* to apply the principle for any central planetary motion, as newtonian dynamics does, though actually by observations, we can not yet ascertain any posslble planetary motian given by the dynamics.

When k becomes great, (4) would be impossible. The reason is that (3) belongs to the elliptic motion. For the parabolic and hyperbolic motions, in newtonian dynamics, we have

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}, \tag{5}$$

respectively

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\frac{2m}{r}+\frac{m}{a}. \tag{6}$$

Comparing (2) and (5), we have by the principle of equivalence

$$k^2-1=0, \quad \text{or} \quad k=1. \tag{7}$$

From (2) and (6),

$$k^2-1=\frac{m}{a}, \quad \text{or} \quad k=\sqrt{1+\frac{m}{a}}. \tag{8}$$

We observe that even when (4) be an imaginary equation, yet (8) is real. (6) is their limiting case.

Taking the light for our signal, assume that the velocity of the light-signal is maximum. By analogy with newtonian case, it is natural

to take it ∞. By the last equation of (2), we have $k=\infty$; hence by the second,

$$ds=0. \tag{9}$$

To obtain the equation of motion of the signal, eliminating ds from (2), we have

$$\frac{r^2}{\gamma}\frac{d\theta}{dt}=\frac{h}{k}.$$

If the signal really exist, we must have

$$\lim_{k\to\infty}\frac{h}{k}=l, \ (\textit{finite, determinate}).$$

Or else, by the last equation of (2), the velocity could not be ∞. We have therefore,

$$\left.\begin{aligned}\frac{r^2}{\gamma}\frac{d\theta}{dt}&=l,\\ \frac{1}{\gamma^2}\left(\frac{dr}{dt}\right)^2+\frac{r^2}{\gamma}\left(\frac{d\theta}{dt}\right)^2&=1,\end{aligned}\right\} \tag{10}$$

which can easily be deduced from Fermat's principle

$$\delta\int dt=0,$$

where $ds^2=0$.

Eliminating dt between the equations (10), the path of the light is

$$\left(\frac{du}{d\vartheta}\right)^2=\frac{1}{l^2}-u^2+2mu^3, \qquad u=\frac{1}{r}, \tag{11}$$

and the light does not propagate in a straight line. It seems verified by the famous astronomical observation.

For the special theory of relativity, we put $m=0$, and (10) becomes

$$\left.\begin{aligned}r^2\frac{d\theta}{dt}&=l\\ \left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2&=1\end{aligned}\right\} \tag{12}$$

Hence the velocity of the light is unity and it propagates in a straight line. The *Massbestimmung* (1) becomes

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2.$$

Changing into the rectangular cartesian coordinates (x, y, z) and the unit of the time, we have

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2, \tag{13}$$

where c is the velocity of the light in vacuum. For the light-signal, we have by (9),

$$dt^2 = \frac{dx^2 + dy^2 + dz^2}{c^2}.$$

We arrived at this result under the assumption that the light-signal has the maximum velocity. In newtonian dynamics, we consider the signal of ∞ —velocity but not of the light. For this case, tending $c \to \infty$, we have

$$dt = 0. \tag{14}$$

Since in newtonian dynamics, there is only one kind of the time, this equation just corresponds to (9) in Einstein's theory. From these considerations, it seems unnecessary to reject ∞ —*velocity* (with respect to the proper time) in Einstein's theory.

Projective Differential Geometry of Nondevelopable Surfaces. I.

By

Jôyô Kanitani.

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CHAPTER I.

INVARIANTS.

1. Let $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$, $\mathfrak{X}_4$ be homogeneous coordinates of the point $\mathfrak{X}$ in the projective space of the three dimensions and suppose that a surface is given by the equations

$$\mathfrak{X}_i = \mathfrak{X}_i(u, v), \qquad (i = 1, 2, 3. 4)$$

where $\mathfrak{X}_i(u, v)$ are analytic functions of u, v in a domain R.

Consider the determinants of the matrix

$$\left\| \begin{array}{ccccccc} \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial u} & \frac{\partial \mathfrak{X}_1}{\partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} \\ \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial \mathfrak{X}_2}{\partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} \\ \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial u} & \frac{\partial \mathfrak{X}_3}{\partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial u^2} & \frac{\partial^3 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} \\ \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial u} & \frac{\partial \mathfrak{X}_4}{\partial v} & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_4}{\partial v^2} \end{array} \right\|$$

and put

$$\left. \begin{aligned} a &= \left| \mathfrak{X}_u \quad \mathfrak{X}_v \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{uv} \right|, \\ a' &= \frac{1}{2} \left| \mathfrak{X}_u \quad \mathfrak{X}_v \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv} \right|, \end{aligned} \right\}$$

$$\left.\begin{aligned}
a'' &= |\,\mathfrak{X} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}\,|,\\
b &= |\,\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{uv}\,|,\\
b' &= \frac{1}{2}|\,\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}\,|,\\
b'' &= |\,\mathfrak{X} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{vv}\,|,\\
c &= |\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{uv}\,|,\\
c' &= \frac{1}{2}|\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uu} \quad \mathfrak{X}_{vv}\,|,\\
c'' &= |\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{uv} \quad \mathfrak{X}_{vv}\,|,\\
d &= |\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uu}\,|,\\
d' &= |\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{uv}\,|,\\
d'' &= |\,\mathfrak{X} \quad \mathfrak{X}_{u} \quad \mathfrak{X}_{v} \quad \mathfrak{X}_{vv}\,|,
\end{aligned}\right\} \cdots\cdots\cdots(1)$$

where $|\,\mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uu} \ \mathfrak{X}_{uv}\,|$ denote the determinants the i^{th} row of which consists of $\frac{\partial \mathfrak{X}_i}{\partial u}$, $\frac{\partial \mathfrak{X}_i}{\partial v}$, $\frac{\partial^2 \mathfrak{X}_i}{\partial u^2}$, $\frac{\partial^2 \mathfrak{X}_i}{\partial v^2}$.

We shall call these determinants the *fuudamental determinants* of the given surface.

From (1) we have

$$\frac{\partial d}{\partial v} = b - 2c' + |\,\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uuv}\,|,$$

$$\frac{\partial d'}{\partial u} = -b + |\,\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uuv}\,|,$$

$$\frac{\partial d'}{\partial v} = -c'' + |\,\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uvv}\,|,$$

$$\frac{\partial d''}{\partial u} = -b' + c'' + |\,\mathfrak{X} \ \mathfrak{X}_u \ \mathfrak{X}_v \ \mathfrak{X}_{uvv}\,|.$$

Therefore,

$$\left.\begin{aligned}
\frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} &= 2(b - c'),\\
\frac{\partial d'}{\partial v} - \frac{\partial d''}{\partial u} &= 2(b' - c'').
\end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Consider the determinant

$$\begin{vmatrix}
\mathscr{X}_1 & \frac{\partial \mathscr{X}_1}{\partial u} & \frac{\partial \mathscr{X}_1}{\partial v} & \frac{\partial^2 \mathscr{X}_1}{\partial u^2} & \frac{\partial^2 \mathscr{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_1}{\partial v^2} & 0 & 0 \\
\mathscr{X}_2 & \frac{\partial \mathscr{X}_2}{\partial u} & \frac{\partial \mathscr{X}_2}{\partial u} & \frac{\partial^2 \mathscr{X}_2}{\partial u^2} & \frac{\partial^2 \mathscr{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_2}{\partial v^2} & 0 & 0 \\
\mathscr{X}_3 & \frac{\partial \mathscr{X}_3}{\partial u} & \frac{\partial \mathscr{X}_3}{\partial v} & \frac{\partial^2 \mathscr{X}_3}{\partial u^2} & \frac{\partial^2 \mathscr{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_3}{\partial v^2} & 0 & 0 \\
\mathscr{X}_4 & \frac{\partial \mathscr{X}_4}{\partial u} & \frac{\partial \mathscr{X}_4}{\partial v} & \frac{\partial^2 \mathscr{X}_4}{\partial u^2} & \frac{\partial^2 \mathscr{X}_4}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_4}{\partial v^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_1}{\partial u^2} & \frac{\partial^2 \mathscr{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_1}{\partial v^2} & \mathscr{X}_1 & \frac{\partial \mathscr{X}_1}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_2}{\partial u^2} & \frac{\partial^2 \mathscr{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_2}{\partial v^2} & \mathscr{X}_2 & \frac{\partial \mathscr{X}_2}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_3}{\partial u^2} & \frac{\partial^2 \mathscr{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_3}{\partial v^2} & \mathscr{X}_3 & \frac{\partial \mathscr{X}_3}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_4}{\partial u^2} & \frac{\partial^3 \mathscr{X}_4}{\partial u \partial v} & \frac{\partial^3 \mathscr{X}_4}{\partial v^2} & \mathscr{X}_4 & \frac{\partial \mathscr{X}_4}{\partial v}
\end{vmatrix} \quad \ldots(3)$$

The determinent (3) vanishes identically. For (3) is equal to

$$\begin{vmatrix}
\mathscr{X}_1 & \frac{\partial \mathscr{X}_1}{\partial u} & \frac{\partial \mathscr{X}_1}{\partial v} & 0 & 0 & 0 & -\mathscr{X}_1 & -\frac{\partial \mathscr{X}_1}{\partial v} \\
\mathscr{X}_2 & \frac{\partial \mathscr{X}_2}{\partial u} & \frac{\partial \mathscr{X}_2}{\partial v} & 0 & 0 & 0 & -\mathscr{X}_2 & -\frac{\partial \mathscr{X}_2}{\partial v} \\
\mathscr{X}_3 & \frac{\partial \mathscr{X}_3}{\partial u} & \frac{\partial \mathscr{X}_3}{\partial v} & 0 & 0 & 0 & -\mathscr{X}_3 & -\frac{\partial \mathscr{X}_3}{\partial v} \\
\mathscr{X}_4 & \frac{\partial \mathscr{X}_4}{\partial u} & \frac{\partial \mathscr{X}_4}{\partial v} & 0 & 0 & 0 & -\mathscr{X}_4 & -\frac{\partial \mathscr{X}_4}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_1}{\partial u^2} & \frac{\partial^2 \mathscr{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_1}{\partial v^2} & \mathscr{X}_1 & \frac{\partial \mathscr{X}_1}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_2}{\partial u^2} & \frac{\partial^2 \mathscr{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_2}{\partial v^2} & \mathscr{X}_2 & \frac{\partial \mathscr{X}_2}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_3}{\partial u^2} & \frac{\partial^2 \mathscr{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_3}{\partial v^2} & \mathscr{X}_3 & \frac{\partial \mathscr{X}_3}{\partial v} \\
0 & 0 & 0 & \frac{\partial^2 \mathscr{X}_4}{\partial u^2} & \frac{\partial^2 \mathscr{X}_4}{\partial u \partial v} & \frac{\partial^2 \mathscr{X}_4}{\partial v^2} & \mathscr{X}_4 & \frac{\partial \mathscr{X}_4}{\partial v}
\end{vmatrix}$$

$$= \begin{vmatrix} \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial u} & \frac{\partial \mathfrak{X}_1}{\partial v} & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial u} & \frac{\partial \mathfrak{X}_2}{\partial v} & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial u} & \frac{\partial \mathfrak{X}_3}{\partial v} & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial u} & \frac{\partial \mathfrak{X}_4}{\partial v} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_1}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_1}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_1}{\partial v^2} & \mathfrak{X}_1 & \frac{\partial \mathfrak{X}_1}{\partial v} \\ 0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_2}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_2}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_2}{\partial v^2} & \mathfrak{X}_2 & \frac{\partial \mathfrak{X}_2}{\partial v} \\ 0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_3}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_3}{\partial u \partial v} & \frac{\partial^2 \mathfrak{X}_3}{\partial v^2} & \mathfrak{X}_3 & \frac{\partial \mathfrak{X}_3}{\partial v} \\ 0 & 0 & 0 & \frac{\partial^2 \mathfrak{X}_4}{\partial u^2} & \frac{\partial^2 \mathfrak{X}_4}{\partial u \partial v} & \frac{\partial^3 \mathfrak{X}_4}{\partial v^2} & \mathfrak{X}_4 & \frac{\partial \mathfrak{X}_4}{\partial v} \end{vmatrix}$$

Expanding the last determinant by the method of Laplace we can see that it vanishes identically.

While if we expand (3) by the method of Laplace it becomes

$$db'' - 2d'b' + d''b.$$

Therefore

$$\left.\begin{aligned} db'' - 2d'b' + d''b = 0. \\ \text{Similarly, we have} \\ dc'' - 2d'c' + d''c = 0, \\ da'' - 2d'a' + d''a = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

2. Consider the transformation

$$\overline{\mathfrak{X}} = \lambda \mathfrak{X} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and denote by $\bar{a}$, $\bar{b}$, etc. the determinants obtained by substituting $\overline{\mathfrak{X}}$ instead of $\mathfrak{X}$ in (1).

Then we have

$$
\left.\begin{aligned}
&\bar{d} = \lambda^4 d, \quad \bar{d}' = \lambda^4 d', \quad \bar{d}'' = \lambda^4 d'', \\
&\bar{b} = \lambda^4\left[b + \frac{\partial \log \lambda}{\partial v} d - 2\frac{\partial \log \lambda}{\partial u} d'\right], \\
&\bar{b}' = \lambda^4\left[b' - \frac{\partial \log \lambda}{\partial u} d''\right], \\
&\bar{b}'' = \lambda^4\left[b'' - \frac{\partial \log \lambda}{\partial u} d''\right], \\
&\bar{c} = \lambda^4\left[c - \frac{\partial \log \lambda}{\partial u} d\right], \\
&\bar{c}' = \lambda_4\left[c' - \frac{\partial \log \lambda}{\partial u} d\right], \\
&\bar{c}'' = \lambda^4\left[c'' + \frac{\partial \log \lambda}{\partial u} d'' - 2\frac{\partial \log \lambda}{\partial v} d'\right], \\
&\bar{a} = \lambda^4\left[a + \frac{\partial \log \lambda}{\partial u} b - \frac{\partial \log \lambda}{\partial v} c + \left\{\begin{matrix}\lambda \\ uu\end{matrix}\right\} d' - \left\{\begin{matrix}\lambda \\ uv\end{matrix}\right\} d\right], \\
&\bar{a}' = \lambda^4\left[a' + \frac{\partial \log \lambda}{\partial u} b' - \frac{\partial \log \lambda}{\partial v} c' + \frac{1}{2}\left\{\begin{matrix}\lambda \\ uu\end{matrix}\right\} d'' - \frac{1}{2}\left\{\begin{matrix}\lambda \\ vv\end{matrix}\right\} d\right], \\
&\bar{a}'' = \lambda^4\left[a'' + \frac{\partial \log \lambda}{\partial u} b'' - \frac{\partial \log \lambda}{\partial v} c'' + \left\{\begin{matrix}\lambda \\ uv\end{matrix}\right\} d'' - \left\{\begin{matrix}\lambda \\ vv\end{matrix}\right\} d'\right],
\end{aligned}\right\} \quad \ldots(6)
$$

where

$$
\left\{\begin{matrix}\lambda \\ uu\end{matrix}\right\} = \frac{\partial^2 \log \lambda}{\partial u^2} - \left(\frac{\partial \log \lambda}{\partial u}\right)^2,
$$

$$
\left\{\begin{matrix}\lambda \\ uv\end{matrix}\right\} = \frac{\partial^2 \log \lambda}{\partial u \partial v} - \frac{\partial \log \lambda}{\partial u} \cdot \frac{\partial \log \lambda}{\partial v},
$$

$$
\left\{\begin{matrix}\lambda \\ vv\end{matrix}\right\} = \frac{\partial^2 \log \lambda}{\partial v^2} - \left(\frac{\partial \log \lambda}{\partial v}\right)^2.
$$

Let us assume that the given surface is not developable and has no porabolic point in the portion which correspond to the domain R of u, v, i. e. $d'^2 - dd'' \neq 0$ in the domain R.

Let $r (r > 0)$ and $\theta (0 \leqq \theta < 2\pi)$ be the modules and the arguments of the functions $d'^2 - dd''$ respectively and put

$$
\Delta = \sqrt{r}\, e^{\frac{i\theta}{2}}. \quad \ldots\ldots(7)
$$

Then, by the transformation (5) we have

$$
\bar{\Delta} = \varepsilon \lambda^4 \Delta, \quad \ldots\ldots(8)
$$

where

$$\varepsilon = 1, \text{ or } -1.$$

Let us call x_1, x_2, x_3, x_4 which are defined by the equations

$$x_i = \frac{\mathfrak{X}_i}{\sqrt[4]{\Delta}}, \quad (i = 1, 2, 3, 4) \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

the *normal coordinates* of the point on the given surface, and the fundamental determinants formed with the normal coordinates the *normal fundamental determinants*, and denote them by A, B,....., etc.

Then

$$\left.\begin{aligned}
D &= \frac{d}{\Delta}, \quad D' = \frac{d'}{\Delta}, \quad D'' = \frac{d''}{\Delta}, \\
B &= \frac{b}{\Delta} + \frac{1}{2}\frac{\partial \log \Delta}{\partial u} D' - \frac{1}{4}\frac{\partial \log \Delta}{\partial v} D, \\
B' &= \frac{b'}{\Delta} + \frac{1}{4}\frac{\partial \log \Delta}{\partial u} D'', \\
B'' &= \frac{b''}{\Delta} + \frac{1}{4}\frac{\partial \log \Delta}{\partial v} D'', \\
C &= \frac{c}{\Delta} + \frac{1}{4}\frac{\partial \log \Delta}{\partial u} D, \\
C' &= \frac{c'}{\Delta} + \frac{1}{4}\frac{\partial \log \Delta}{\partial v} D, \\
C'' &= \frac{c''}{\Delta} + \frac{1}{2}\frac{\partial \log \Delta}{\partial v} D' - \frac{1}{4}\frac{\partial \log \Delta}{\partial u} D'', \\
A &= \frac{a}{\Delta} - \frac{1}{4}\frac{\partial \log \Delta}{\partial u} B + \frac{1}{4}\frac{\partial \log \Delta}{\partial v} C + \left\{\begin{matrix} \frac{1}{4}\Delta \\ uv \end{matrix}\right\} D - \left\{\begin{matrix} \frac{1}{4}\Delta \\ uu \end{matrix}\right\} D', \\
A' &= \frac{a'}{\Delta} - \frac{1}{4}\frac{\partial \log \Delta}{\partial u} B' + \frac{1}{4}\frac{\partial \log \Delta}{\partial v} C' + \frac{1}{2}\left\{\begin{matrix} \frac{1}{4}\Delta \\ vv \end{matrix}\right\} D - \frac{1}{2}\left\{\begin{matrix} \frac{1}{4}\Delta \\ uu \end{matrix}\right\} D'', \\
A'' &= \frac{a''}{\Delta} - \frac{1}{4}\frac{\partial \log \Delta}{\partial u} B'' + \frac{1}{4}\frac{\partial \log \Delta}{\partial v} C'' + \left\{\begin{matrix} \frac{1}{4}\Delta \\ vv \end{matrix}\right\} D' - \left\{\begin{matrix} \frac{1}{4}\Delta \\ uv \end{matrix}\right\} D''.
\end{aligned}\right\} \quad (10)$$

We can easily see that the normal fundamental determinants remain unchanged by the transformation of the form (5), or the projective transformation, except for a sign.

Evidently, we have

$$\left.\begin{aligned}
&\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}=2(B-C'),\\
&\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}=2(B'-C''),\\
&DB''-2D'B'+D''B=0,\\
&DC''-2D'C'+D''C=0,\\
&DA''-2D'A'+D''A=0
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

and, moreover, we have

$$D'^2-DD''=1.$$

Differentiating this equation by u and by v, we have

$$\left.\begin{aligned}
&D\frac{\partial D''}{\partial u}-2D'\frac{\partial D'}{\partial u}+D''\frac{\partial D}{\partial u}=0,\\
&D\frac{\partial D''}{\partial v}-2D'\frac{\partial D'}{\partial v}+D''\frac{\partial D}{\partial v}=0.
\end{aligned}\right\}\quad\cdots\cdots\cdots(12)$$

3. Next, let us consider the transformation

$$\left.\begin{aligned}
&u=u(\xi,\ \eta),\\
&v=v(\xi,\ \eta),
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots(13)$$

where

$$W=\frac{\partial(u,\ v)}{\partial(\xi,\ \eta)}\neq 0.$$

Put

$$\frac{\partial u}{\partial\xi}=p,\quad \frac{\partial u}{\partial\eta}=q,\quad \frac{\partial^2 u}{\partial\xi^2}=r,\quad \frac{\partial^2 u}{\partial\xi\partial\eta}=s,\quad \frac{\partial^2 u}{\partial\eta^2}=t,$$

$$\frac{\partial v}{\partial\xi}=p_1,\quad \frac{\partial v}{\partial\eta}=q_1,\quad \frac{\partial^2 v}{\partial\xi^2}=r_1,\quad \frac{\partial^2 v}{\partial\xi\partial\eta}=s_1,\quad \frac{\partial^2 v}{\partial\eta^2}=t_1.$$

Then

$$\left.\frac{\partial x}{\partial\xi}=\frac{\partial x}{\partial u}p+\frac{\partial x}{\partial v}p_1,\right.$$

$$\left.\begin{aligned}
&\frac{\partial x}{\partial \eta}=\frac{\partial x}{\partial u}q+\frac{\partial x}{\partial v}q_1,\\
&\frac{\partial^2 x}{\partial \xi^2}=\frac{\partial^2 x}{\partial u^2}p^2+2\frac{\partial^2 x}{\partial u\partial v}pp_1+\frac{\partial^2 x}{\partial v^2}p_1^2+\frac{\partial x}{\partial u}r+\frac{\partial x}{\partial v}r_1,\\
&\frac{\partial^2 x}{\partial \xi\partial \eta}=\frac{\partial^2 x}{\partial u^2}pq+\frac{\partial^2 x}{\partial u\partial v}(pq_1+p_1q)+\frac{\partial^2 x}{\partial v^2}p_1q_1+\frac{\partial x}{\partial u}s+\frac{\partial x}{\partial v}s_1,\\
&\frac{\partial^2 x}{\partial \eta^2}=\frac{\partial^2 x}{\partial u^2}q^2+2\frac{\partial^2 x}{\partial u\partial v}qq_1+\frac{\partial^2 x}{\partial v^2}q_1^2+\frac{\partial x}{\partial u}t+\frac{\partial x}{\partial v}t_1.
\end{aligned}\right\}\quad(14)$$

Denote by $\bar{a}$, $\bar{b}$, etc. the fundamental determinants referred to the parameters ξ, η, i. e. the determinants obtained by substituting in (1) $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial^2 x}{\partial \xi^2}$, $\frac{\partial^2 x}{\partial \xi\partial \eta}$, $\frac{\partial^2 x}{\partial \eta^2}$ instead of $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial^2 x}{\partial u^2}$, $\frac{\partial^2 x}{\partial u\partial v}$, $\frac{\partial^2 x}{\partial v}$ respectively.

Then we have from (1) and (14)

$$\bar{d}=W\Sigma p^2d,\quad \bar{d}'=W\Sigma pqd,\quad \bar{d}''=W\Sigma q^2d,\ \ldots\ldots(15)$$

where

$$\Sigma p^2d=p^2d+2pp_1d'+p_1^2d'',$$

$$\Sigma pqd=pqd+(pq_1+p_1q)d'+p_1q_1d'',$$

$$\Sigma q^2d=q^2d+2qq_1d'+q_1^2d''.$$

From (15) we have

$$\bar{\Delta}=\varepsilon W^2\Delta,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

where $\varepsilon=+1$, or -1.

Therefore we have

$$\bar{D}=\varepsilon\Sigma\frac{p^2}{W}D,\quad \bar{D}'=\varepsilon\Sigma\frac{pq}{W}D',\quad \bar{D}''=\varepsilon\Sigma\frac{q^2}{W}D.\ \ldots.(17)$$

Let $\mathfrak{T}$, $\mathfrak{T}'$, $\mathfrak{T}''$ be three independent variables. Then we have the following identities which are very useful in the subsequent calculations.

$$\left.\begin{aligned}
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}=\left(\begin{vmatrix} r & q \\ r_1 & q_1 \end{vmatrix}-\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}\right)\Sigma\frac{p^2}{W^2}\mathfrak{T}+2\begin{vmatrix} p & r \\ p_1 & r_1 \end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)\mathfrak{T}=\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\begin{vmatrix} p & r \\ p_1 & r_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)\mathfrak{T}=2\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}-\begin{vmatrix} r & q \\ r_1 & q_1 \end{vmatrix}\right)\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)\mathfrak{T}=\left(\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}-\begin{vmatrix} p & t \\ p_1 & t_1 \end{vmatrix}\right)\Sigma\frac{p^2}{W^2}\mathfrak{T}+2\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{pq}{W}\right)\mathfrak{T}=\begin{vmatrix} t & q \\ t_1 & q_1 \end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T},\\
&\Sigma\frac{\partial}{\partial\eta}\left(\frac{q^2}{W}\right)\mathfrak{T}=2\begin{vmatrix} t & q \\ t_1 & q_1 \end{vmatrix}\Sigma\frac{pq}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & t \\ p_1 & t_1 \end{vmatrix}-\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\right)\Sigma\frac{q}{W^2}\mathfrak{T},
\end{aligned}\right\}\quad(18)$$

where

$$\Sigma\frac{p^2}{W}\mathfrak{T}=\frac{p^2}{W}\mathfrak{T}+\frac{2pp_1}{W}\mathfrak{T}'+\frac{p_1^2}{W}\mathfrak{T}'',$$

$$\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}=\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)\mathfrak{T}+2\frac{\partial}{\partial\xi}\left(\frac{pp_1}{W}\right)\mathfrak{T}'+\frac{\partial}{\partial\xi}\left(\frac{p_1^2}{W}\right)\mathfrak{T}'',$$

etc.

From (1) and (14) we have

$$\bar{b}=qW\Sigma p^2c+q_1W\Sigma p^2b+\begin{vmatrix} q & r \\ q_1 & r_1 \end{vmatrix}\Sigma pqd+\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\Sigma p^2d.$$

Therefore, we have from (9) and (18)

$$\begin{aligned}
\bar{B}=\varepsilon\Big[&q\Sigma\left(\frac{p^2}{W}C\right)+q_1\left(\Sigma\frac{p^2}{W}B\right)\\
&+\left(\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}-\begin{vmatrix} p & t \\ p_1 & t_1 \end{vmatrix}\right)\Sigma\frac{p^2}{W^2}D+\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}\Sigma\frac{pq}{W^2}D\Big]\\
=\varepsilon\Big[&q\left(\Sigma\frac{p^2}{W}C\right)+q_1\left(\Sigma\frac{p^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{p^2}{W}\right)D\Big].
\end{aligned}$$

Similarly we have

$$\bar{B}'=\varepsilon\Big[q\left(\Sigma\frac{pq}{W}C\right)+q_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{pq}{W}\right)D\Big],$$

$$\overline{B''}=\varepsilon\left[q\left(\Sigma\frac{q^2}{W}C\right)+q_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\left(\frac{q^2}{W}\right)D\right],$$

$$\overline{C}=\varepsilon\left[p\left(\Sigma\frac{p^2}{W}C\right)+p_1\left(\Sigma\frac{p^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\right],$$

$$\overline{C'}=\varepsilon\left[p\left(\Sigma\frac{pq}{W}C\right)+p_1\left(\Sigma\frac{pq}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D\right],$$

$$\overline{C''}=\varepsilon\left[p\left(\Sigma\frac{q^2}{W}C\right)+p_1\left(\Sigma\frac{q^2}{W}B\right)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D\right].$$

Therefore, if we put

$$\left.\begin{aligned}
\mathfrak{B}&=B-\frac{1}{2}\frac{\partial D}{\partial v}, & \mathfrak{C}&=C-\frac{1}{2}\frac{\partial D}{\partial u},\\
\mathfrak{B}'&=B'-\frac{1}{2}\frac{\partial D'}{\partial v}, & \mathfrak{C}'&=C'-\frac{1}{2}\frac{\partial D'}{\partial u},\\
\mathfrak{B}''&=B''-\frac{1}{2}\frac{\partial D''}{\partial v}, & \mathfrak{C}''&=C''-\frac{1}{2}\frac{\partial D''}{\partial u},
\end{aligned}\right\}\quad\cdots\cdots\cdots(19)$$

we have

$$\left.\begin{aligned}
\overline{\mathfrak{B}}&=\varepsilon\left[q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right],\\
\overline{\mathfrak{B}'}&=\varepsilon\left[q\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right],\\
\overline{\mathfrak{B}''}&=\varepsilon\left[q\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right],\\
\overline{\mathfrak{C}}&=\varepsilon\left[p\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right],\\
\overline{\mathfrak{C}'}&=\varepsilon\left[p\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right],\\
\overline{\mathfrak{C}''}&=\varepsilon\left[p\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right].
\end{aligned}\right\}\quad\cdots\cdots\cdots(20)$$

From (11) and (12) we have

$$\left.\begin{aligned}
D\mathfrak{B}''-2\mathfrak{B}'D'+D''\mathfrak{B}&=0,\\
D\mathfrak{C}''-2\mathfrak{C}'D'+D''\mathfrak{C}&=0.
\end{aligned}\right\}\quad\cdots\cdots\cdots(21)$$

Therefore we have

$$\frac{2\begin{vmatrix}\mathfrak{C} & \mathfrak{B}\\ \mathfrak{C}' & \mathfrak{B}'\end{vmatrix}}{D}=\frac{\begin{vmatrix}\mathfrak{C} & \mathfrak{B}\\ \mathfrak{C}'' & \mathfrak{B}''\end{vmatrix}}{D'}=\frac{2\begin{vmatrix}\mathfrak{C}' & \mathfrak{B}'\\ \mathfrak{C}'' & \mathfrak{B}''\end{vmatrix}}{D''}=|\mathfrak{B}\ \mathfrak{C}\ D|, \quad \ldots(22)$$

where

$$|\mathfrak{B}\ \mathfrak{C}\ D|=\begin{vmatrix}\mathfrak{B} & \mathfrak{C} & D\\ \mathfrak{B}' & \mathfrak{C}' & D'\\ \mathfrak{B}'' & \mathfrak{C}'' & D''\end{vmatrix},$$

for $$D'^2-DD''=1.$$

We shall denote by θ the last determinant.

We can see from (17) and (20) that θ is a relative invariant which satisfies the equation

$$\bar{\theta}=\varepsilon W\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

We shall prove, afterwards, that θ does not vanish identically for a curved surface.

From (11) and (19) we have

$$\mathfrak{B}=\mathfrak{C}', \quad \mathfrak{B}'=\mathfrak{C}''.$$

From these equations and (20) we know that

$$\mathfrak{C}du^3+3\mathfrak{B}du^2dv+3\mathfrak{C}''dudv^2+\mathfrak{B}''dv^3=0,$$

is an invariant equation. We shall show, afterwards, that this is the equation of the Darboux's curves of quadric osculation.

From (1) and (13) we have

$$\bar{a}=W^2(\Sigma p^2 a).$$

Therefore, we have from (11)

$$\bar{A}=\varepsilon\Big[\Sigma p^2A-\frac{1}{2}\frac{\partial\log W}{\partial\xi}\Big\{q\Big(\Sigma\frac{p^2}{W}C\Big)+q_1\Big(\Sigma\frac{p^2}{W}B\Big)+\frac{1}{2}\Sigma\frac{\partial}{\partial\eta}\Big(\frac{p^2}{W}\Big)D\Big\}$$

$$+\frac{1}{2}\frac{\partial\log W}{\partial\eta}\Big\{p\Big(\Sigma\frac{p^2}{W}C\Big)+p_1\Big(\Sigma\frac{p^2}{W}B\Big)+\frac{1}{2}\Sigma\frac{\partial}{\partial\xi}\Big(\frac{p^2}{W}\Big)D\Big\}$$

$$+\Big\{\frac{\frac{1}{2}W}{\xi\eta}\Big\}\Sigma\frac{p^2}{W}D-\Big\{\frac{\frac{1}{2}W}{\xi\xi}\Big\}\Sigma\frac{pq}{W}D\Big].$$

Similarly we have

$$\bar{A}' = \varepsilon\Big[\Sigma pq A - \frac{1}{2}\frac{\partial \log W}{\partial \xi}\Big\{q\Big(\Sigma \frac{pq}{W} C\Big) + q_1\Big(\Sigma \frac{pq}{W} B\Big) + \frac{1}{2}\Sigma \frac{\partial}{\partial \eta}\Big(\frac{pq}{W}\Big) D\Big\}$$

$$+ \frac{1}{2}\frac{\partial \log W}{\partial \eta}\Big\{p\Big(\Sigma \frac{pq}{W} C\Big) + p_1\Big(\Sigma \frac{pq}{W} B\Big) + \frac{1}{2}\Sigma \frac{\partial}{\partial \xi}\Big(\frac{pq}{W}\Big) D\Big\}$$

$$+ \frac{1}{2}\Big\{\frac{\frac{1}{2}W}{\eta\eta}\Big\}\Sigma \frac{p^2}{W} D - \frac{1}{2}\Big\{\frac{\frac{1}{2}W}{\xi\xi}\Big\}\Sigma \frac{q^2}{W} D\Big],$$

$$\bar{A}'' = \varepsilon\Big[\Sigma q^2 A - \frac{1}{2}\frac{\partial \log W}{\partial \xi}\Big\{q\Big(\Sigma \frac{q^2}{W} C\Big) + q_1\Big(\Sigma \frac{q^2}{W} B\Big) + \frac{1}{2}\Sigma \frac{\partial}{\partial \eta}\Big(\frac{q^2}{W}\Big) D\Big\}$$

$$+ \frac{1}{2}\frac{\partial \log W}{\partial \eta}\Big\{p\Big(\Sigma \frac{q^2}{W} C\Big) + q_1\Big(\Sigma \frac{q^2}{W} B\Big) + \frac{1}{2}\Sigma \frac{\partial}{\partial \xi}\Big(\frac{q^2}{W}\Big) D\Big\}$$

$$+ \Big\{\frac{\frac{1}{2}W}{\eta\eta}\Big\}\Sigma \frac{pq}{W} D - \Big\{\frac{\frac{1}{2}W}{\xi\xi}\Big\}\Sigma \frac{q^2}{W} D\Big].$$

Therefore, if we put

$$\left.\begin{aligned}
\mathfrak{A} &= A + \frac{1}{2}\frac{\partial \log \theta}{\partial u} B - \frac{1}{2}\frac{\partial \log \theta}{\partial v} C + \Big\{\frac{\frac{1}{2}\theta}{uu}\Big\} D' - \Big\{\frac{\frac{1}{2}\theta}{uv}\Big\} D, \\
\mathfrak{A}' &= A' + \frac{1}{2}\frac{\partial \log \theta}{\partial u} B' - \frac{1}{2}\frac{\partial \log \theta}{\partial v} C' + \frac{1}{2}\Big\{\frac{\frac{1}{2}\theta}{uu}\Big\} D'' - \frac{1}{2}\Big\{\frac{\frac{1}{2}\theta}{vv}\Big\} D, \\
\mathfrak{A}'' &= A'' + \frac{1}{2}\frac{\partial \log \theta}{\partial u} B'' - \frac{1}{2}\frac{\partial \log \theta}{\partial v} C'' + \Big\{\frac{\frac{1}{2}\theta}{uv}\Big\} D'' - \Big\{\frac{\frac{1}{2}\theta}{vv}\Big\} D',
\end{aligned}\right\} \quad (24)$$

we have

$$\left.\begin{aligned}
\bar{\mathfrak{A}} &= \varepsilon[\Sigma p^2 \mathfrak{A}], \\
\bar{\mathfrak{A}}' &= \varepsilon[\Sigma pq \mathfrak{A}], \\
\bar{\mathfrak{A}}'' &= \varepsilon[\Sigma q^2 \mathfrak{A}],
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

From (25) we know that $\mathfrak{A}\mathfrak{A}'' - \mathfrak{A}'^2$ is a relative invariant and

$$\mathfrak{A} du^2 + 2\mathfrak{A}' du dv + \mathfrak{A}'' dv^2 = 0,$$

is an invariant equation. Afterwards, we shall interpretate them geometrically.

From (11) and (12) we have

$$D\mathfrak{A}'' - 2D'\mathfrak{A}' + D''\mathfrak{A} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

If the number ε in the equation (16) is equal to -1, then let us carry out the transformation

$$\overline{\mathfrak{X}} = e^{\frac{i\pi}{4}} \mathfrak{X}.$$

again. By this transformation $\bar{\Delta}$ remains unchanged by its definition, but the signs of $\overline{D}$, $\overline{D}$, $\overline{D}''$; $\overline{\mathfrak{B}}$, $\overline{\mathfrak{B}}$, $\overline{\mathfrak{B}}''$; $\overline{\mathfrak{C}}$, $\overline{\mathfrak{C}}'$, $\overline{\mathfrak{C}}''$; $\overline{\mathfrak{A}}$, $\overline{\mathfrak{A}}'$, $\overline{\mathfrak{A}}''$ are changed.

Therefore, we may suppose the number ε in the equations (17), (20), (23) and (25) to be equal to $+1$.

4. From (20) we have

$$\begin{aligned}\frac{\partial \overline{\mathfrak{B}}}{\partial \xi} - \frac{\partial \overline{\mathfrak{C}}}{\partial \eta} = {} & q\left\{\Sigma \frac{\partial}{\partial \xi}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\} + q_1\left\{\Sigma \frac{\partial}{\partial \xi}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\} \\ & - p\left\{\Sigma \frac{\partial}{\partial \eta}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\} - p_1\left\{\Sigma \frac{\partial}{\partial \eta}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\} \\ & + \Sigma p^2\left(\frac{\partial \mathfrak{B}}{\partial u} - \frac{\partial \mathfrak{B}}{\partial v}\right).\end{aligned}$$

And from the relation

$$\mathfrak{B} = \mathfrak{C}', \quad \mathfrak{B}' = \mathfrak{C}'',$$

we have

$$q\left(\Sigma \frac{p^2}{W}\mathfrak{C}\right) + q_1\left(\Sigma \frac{p^2}{W}\mathfrak{B}\right) = p\left(\Sigma \frac{pq}{W}\mathfrak{C}\right) + p_1\left(\Sigma \frac{pq}{W}\mathfrak{B}\right),$$

$$q\left(\Sigma \frac{pq}{W}\mathfrak{C}\right) + q_1\left(\Sigma \frac{pq}{W}\mathfrak{B}\right) = p\left(\Sigma \frac{q^2}{W}\mathfrak{C}\right) + p_1\left(\Sigma \frac{q^2}{W}\mathfrak{B}\right).$$

In virtue of the last two equations and (19), we have

$$\begin{aligned}& q\left\{\Sigma \frac{\partial}{\partial \xi}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\} + q_1\left\{\Sigma \frac{\partial}{\partial \xi}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\} - p\left\{\Sigma \frac{\partial}{\partial \eta}\left(\frac{p^2}{W}\right)\mathfrak{C}\right\} - p_1\left\{\Sigma \frac{\partial}{\partial \eta}\left(\frac{p^2}{W}\right)\mathfrak{B}\right\} \\ & = -\frac{\partial \log W}{\partial \xi}\left\{q\left(\Sigma \frac{p^2}{W}\mathfrak{C}\right) + q_1\left(\Sigma \frac{p^2}{W}\mathfrak{B}\right)\right\} \\ & + \frac{\partial \log W}{\partial \eta}\left\{p\left(\Sigma \frac{p^2}{W}\mathfrak{C}\right) + p_1\left(\Sigma \frac{p^2}{W}\mathfrak{B}\right) - 2p\varphi(\mathfrak{C}) - 2p_1\varphi(\mathfrak{B}),\right.\end{aligned}$$

where

$$\varphi(\mathfrak{T})=\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & s \\ p_1 & s_1 \end{vmatrix}-\begin{vmatrix} r & q \\ r_1 & q_1 \end{vmatrix}\right)\Sigma\frac{pq}{W^2}\mathfrak{T}+\begin{vmatrix} r & p \\ r_1 & p_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T}.$$

Therefore, we have

$$\left.\begin{aligned}
\frac{\partial\overline{\mathfrak{B}}}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}}{\partial\eta}&=-\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right\}\\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)+p_1\left(\Sigma\frac{p^2}{W}\mathfrak{B}\right)\right\}\\
&\quad-2p\varphi(\mathfrak{C})-2p_1\varphi(\mathfrak{B})+\Sigma p^2\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right).\\
\frac{\partial\overline{\mathfrak{B}}'}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}'}{\partial\eta}&=-\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right\}\\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{pq}{W}\mathfrak{B}\right)\right\}\\
&\quad-q\varphi(\mathfrak{C})-q_1\varphi(\mathfrak{B})-p\psi(\mathfrak{C})-p_1\psi(\mathfrak{B})+\Sigma pq\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right),\\
\frac{\partial\overline{\mathfrak{B}}''}{\partial\xi}-\frac{\partial\overline{\mathfrak{C}}''}{\partial\eta}&=-\frac{\partial\log W}{\partial\xi}\left\{q\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right\}\\
&\quad+\frac{\partial\log W}{\partial\eta}\left\{p\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)+q_1\left(\Sigma\frac{q^2}{W}\mathfrak{B}\right)\right\}\\
&\quad-2q\psi(\mathfrak{C})-2q_1\psi(\mathfrak{B})+\Sigma q^2\left(\frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}\right),
\end{aligned}\right\}\quad(27)$$

where

$$\psi(\mathfrak{T})=\begin{vmatrix} t & q \\ t_1 & q_1 \end{vmatrix}\Sigma\frac{p^2}{W^2}\mathfrak{T}+\left(\begin{vmatrix} p & t \\ p_1 & t_1 \end{vmatrix}-\begin{vmatrix} s & q \\ s_1 & q_1 \end{vmatrix}\right)\Sigma\frac{pq}{W^2}\mathfrak{T}+\begin{vmatrix} s & p \\ s_1 & p_1 \end{vmatrix}\Sigma\frac{q^2}{W^2}\mathfrak{T}$$

From (12) and (20) we have

$$\left.\frac{2\begin{vmatrix} D_u & \mathfrak{C} \\ D_u' & \mathfrak{C}' \end{vmatrix}}{D}=\frac{\begin{vmatrix} D_u & \mathfrak{C} \\ D_u' & \mathfrak{C}'' \end{vmatrix}}{D'}=\frac{2\begin{vmatrix} D_u' & \mathfrak{C}' \\ D_u'' & \mathfrak{C}'' \end{vmatrix}}{D''}=|\mathfrak{C}\ D_u\ D|,\right\}$$

$$\left.\begin{aligned}
&\frac{2\begin{vmatrix} D_u & \mathfrak{B} \\ D'_u & \mathfrak{B}' \end{vmatrix}}{D} = \frac{\begin{vmatrix} D_u & \mathfrak{B} \\ D''_u & \mathfrak{B}'' \end{vmatrix}}{D'} = \frac{2\begin{vmatrix} D'_u & \mathfrak{B}' \\ D''_u & \mathfrak{B}'' \end{vmatrix}}{D''} = |\mathfrak{B}\, D_u\, D|,\\
&\frac{2\begin{vmatrix} D_v & \mathfrak{C} \\ D'_v & \mathfrak{C}' \end{vmatrix}}{D} = \frac{\begin{vmatrix} D_v & \mathfrak{C} \\ D''_v & \mathfrak{C}'' \end{vmatrix}}{D'} = \frac{2\begin{vmatrix} D'_v & \mathfrak{C}' \\ D''_v & \mathfrak{C}'' \end{vmatrix}}{D''} = |\mathfrak{C}\, D_v\, D|,\\
&\frac{2\begin{vmatrix} D_v & \mathfrak{B} \\ D'_v & \mathfrak{B}' \end{vmatrix}}{D} = \frac{\begin{vmatrix} D_v & \mathfrak{B} \\ D''_v & \mathfrak{B}'' \end{vmatrix}}{D'} = \frac{2\begin{vmatrix} D'_v & \mathfrak{B}' \\ D''_v & \mathfrak{B}'' \end{vmatrix}}{D''} = |\mathfrak{B}\, D_v\, D|.
\end{aligned}\right\} \quad (28)$$

From (17) and (20) we have

$$\begin{aligned}
|\overline{\mathfrak{C}}\ \overline{D}_\xi\ \overline{D}| = {} & p\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\ \ \Sigma\frac{p^2}{W}D|\\
& + p_1\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D\ \ \Sigma\frac{p^2}{W}D|\\
& + p^2\,|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{p^2}{W}D_u\ \ \Sigma\frac{p^2}{W}D|\\
& + pp_1\{|\Sigma\frac{p^2}{W}\mathfrak{C}\ \ \Sigma\frac{p^2}{W}D_v\ \ \Sigma\frac{p^2}{W}D|+|\Sigma\frac{p^2}{W}\mathfrak{B}\ \Sigma\frac{p^2}{W}D_u\ \Sigma\frac{p^2}{W}D|\}\\
& + p_1^2|\Sigma\frac{p^2}{W}\mathfrak{B}\ \Sigma\frac{p^2}{W}D_v\ \Sigma\frac{p^2}{W}D|.
\end{aligned}$$

But in virtue of (19) and the equation

$$\left(\Sigma\frac{p^2}{W}D\right)\left(\Sigma\frac{q^2}{W}\mathfrak{C}\right)-2\left(\Sigma\frac{pq}{W}D\right)\left(\Sigma\frac{pq}{W}\mathfrak{C}\right)+\left(\Sigma\frac{q^2}{W}D\right)\left(\Sigma\frac{p^2}{W}\mathfrak{C}\right)=0,$$

which follows from the equation

$$D\mathfrak{C}''-2D\mathfrak{C}'+D''\mathfrak{C}=0,$$

we have

$$(\Sigma\frac{p^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D)-(\Sigma\frac{pq}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{p^2}{W}\right)D)=(\Sigma\frac{p^2}{W}D)\varphi(\mathfrak{C}),$$

$$(\Sigma\frac{p^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D)-(\Sigma\frac{q^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\ \frac{p^2}{W}D)=2(\Sigma\frac{pq}{W}D)\varphi(\mathfrak{C}),$$

$$(\Sigma\frac{pq}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{q^2}{W}\right)D)-(\Sigma\frac{q^2}{W}\mathfrak{C})(\Sigma\frac{\partial}{\partial\xi}\left(\frac{pq}{W}\right)D)=(\Sigma\frac{q^2}{W}D)\varphi(\mathfrak{C}).$$

The last three equations also hold if we substitute $\mathfrak{B}$ for $\mathfrak{C}$. Therefore, we have

$$|\overline{\mathfrak{C}}\ \overline{D}_\xi\ \overline{D}| = -2p\varphi(\mathfrak{C})-2p_1\varphi(\mathfrak{B})+p^2|\mathfrak{C}\ D_u\ D|$$
$$+pp_1\{|\mathfrak{C}\ D_v\ D|+|\mathfrak{B}\ D_u\ D|\}+p_1^2|\mathfrak{B}\ D_v\ D|.$$

Similarly, we have

$$\tfrac{1}{2}\{|\overline{\mathfrak{C}}\ \overline{D}_\eta\ \overline{D}|+|\overline{\mathfrak{B}}\ \overline{D}_\xi\ D|\}$$
$$=-p\psi(\mathfrak{C})-p_1\psi(\mathfrak{B})-q\varphi(\mathfrak{C})-q_1\varphi(\mathfrak{B})$$
$$+pq|\mathfrak{C}\ D_u\ D|+\frac{pq_1+p_1q}{2}\{|\mathfrak{C}\ D_v\ D|+|\mathfrak{B}\ D_u\ D|\}$$
$$+p_1q_1|\mathfrak{B}\ D_v\ D|,$$

$$|\overline{\mathfrak{B}}\ \overline{D}_\eta\ \overline{D}| = -2q\psi(\mathfrak{C})-2q_1\psi(\mathfrak{B})+q^2|\mathfrak{C}\ D_w\ D|$$
$$+qq_1\{|\mathfrak{C}\ D_v\ D|+|\mathfrak{B}\ D_u\ D|\}+q_1^2|\mathfrak{B}\ D_v\ D|.$$

Therefore, if we put

$$\left.\begin{aligned} L &= \frac{\partial\mathfrak{B}}{\partial u}-\frac{\partial\mathfrak{C}}{\partial v}-|\mathfrak{C}\ D_u\ D|,\\ L' &= \frac{\partial\mathfrak{B}'}{\partial u}-\frac{\partial\mathfrak{C}'}{\partial v}-\frac{1}{2}\{|\mathfrak{C}\ D_v\ D|+|\mathfrak{B}\ D_u\ D|\}\\ L'' &= \frac{\partial\mathfrak{B}''}{\partial u}-\frac{\partial\mathfrak{C}''}{\partial v}-|\mathfrak{B}\ D_v\ D|, \end{aligned}\right\}\quad(29)$$

$$\left.\begin{aligned} \mathfrak{L} &= L+\frac{\partial\log\theta}{\partial u}\mathfrak{B}-\frac{\partial\log\theta}{\partial v}\mathfrak{C},\\ \mathfrak{L}' &= L'+\frac{\partial\log\theta}{\partial u}\mathfrak{B}'-\frac{\partial\log\theta}{\partial v}\mathfrak{C}',\\ \mathfrak{L}'' &= L''+\frac{\partial\log\theta}{\partial u}\mathfrak{B}''-\frac{\partial\log\theta}{\partial v}\mathfrak{C}'' \end{aligned}\right\}\quad(30)$$

we have

$$\left.\begin{aligned}\overline{\mathfrak{L}} &= \Sigma p^2 \mathfrak{L},\\ \overline{\mathfrak{L}}' &= \Sigma pq \mathfrak{L},\\ \overline{\mathfrak{L}}'' &= \Sigma q^2 \mathfrak{L}.\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

From (28) we have

$$D''\,|\,\mathfrak{C}\,D_u D\,| = 2\left(\mathfrak{C}''\frac{\partial D'}{\partial u} - \mathfrak{C}'\frac{\partial D''}{\partial u}\right) = 2\left(\mathfrak{B}'\frac{\partial D'}{\partial u} - \mathfrak{B}\frac{\partial D''}{\partial u}\right),$$

$$D'\{\,|\,\mathfrak{C}\,D_v D\,| + |\,\mathfrak{B}\,D_u D\,|\,\}$$

$$= \mathfrak{B}''\frac{\partial D}{\partial u} - \mathfrak{B}\frac{\partial D''}{\partial u} + \mathfrak{C}''\frac{\partial D}{\partial v} - \mathfrak{C}\frac{\partial D''}{\partial v},$$

$$D\,|\,\mathfrak{B}\,D_v D\,| = 2\left(\mathfrak{B}'\frac{\partial D}{\partial v} - \mathfrak{B}\frac{\partial D'}{\partial v}\right) = 2\left(\mathfrak{C}''\frac{\partial D}{\partial v} - \mathfrak{C}'\frac{\partial D'}{\partial v}\right).$$

Hence, we have

$$DL'' - 2D'L' + D''L$$

$$= \frac{\partial}{\partial u}(D\mathfrak{B}'' - 2D'\mathfrak{B}' + D''\mathfrak{B})$$

$$- \frac{\partial}{\partial v}(D\mathfrak{C}' - 2D'\mathfrak{C}' + D''\mathfrak{C}) = 0,$$

$$D\mathfrak{L}'' + 2D'\mathfrak{L}' + D''\mathfrak{L}$$

$$= DL'' - 2D'L' + D''L$$

$$+ \frac{\partial \log\theta}{\partial u}(D\mathfrak{B}'' - 2D'\mathfrak{B}' + D''\mathfrak{B})$$

$$- \frac{\partial \log\theta}{\partial v}(D\mathfrak{C}'' - 2D'\mathfrak{C}' + D''\mathfrak{C}) = 0.$$

Therefore, we have

$$\left.\begin{aligned}\frac{2\begin{vmatrix}\mathfrak{C} & \mathfrak{L}\\ \mathfrak{C}' & \mathfrak{L}'\end{vmatrix}}{D} &= \frac{\begin{vmatrix}\mathfrak{C} & \mathfrak{L}\\ \mathfrak{C}'' & \mathfrak{L}''\end{vmatrix}}{D'} = \frac{2\begin{vmatrix}\mathfrak{C}' & \mathfrak{L}'\\ \mathfrak{C}'' & \mathfrak{L}''\end{vmatrix}}{D''} = |\,\mathfrak{L}\ \mathfrak{C}\ D\,|,\\ \frac{2\begin{vmatrix}\mathfrak{B} & \mathfrak{L}\\ \mathfrak{B}' & \mathfrak{L}'\end{vmatrix}}{D} &= \frac{\begin{vmatrix}\mathfrak{B} & \mathfrak{L}\\ \mathfrak{B}'' & \mathfrak{L}''\end{vmatrix}}{D'} = \frac{2\begin{vmatrix}\mathfrak{B}' & \mathfrak{L}'\\ \mathfrak{B}'' & \mathfrak{L}''\end{vmatrix}}{D''} = |\,\mathfrak{L}\ \mathfrak{B}\ D\,|.\end{aligned}\right\} \ldots (32)$$

From (17), (20) and (31) we have

$$\left.\begin{aligned} |\overline{\mathfrak{L}}\ \overline{\mathfrak{C}}\ \overline{D}| &= pW|\mathfrak{L}\ \mathfrak{C}\ D| + p_1W|\mathfrak{L}\ \mathfrak{B}\ D|, \\ |\overline{\mathfrak{L}}\ \overline{\mathfrak{B}}\ \overline{D}| &= qW|\mathfrak{L}\ \mathfrak{C}\ D| + q_1W|\mathfrak{L}\ \mathfrak{B}\ D|. \end{aligned}\right\} \quad \cdots\cdots\cdots(33)$$

5. From (24) we have

$$\frac{\partial\overline{\mathfrak{A}}}{\partial\eta} - \frac{\partial\overline{\mathfrak{A}}'}{\partial\xi} = \Sigma\frac{\partial}{\partial\eta}(p^2)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(pq)\mathfrak{A} + pW\left(\frac{\partial\mathfrak{A}}{\partial v} - \frac{\partial\mathfrak{A}'}{\partial u}\right) + p_1W\left(\frac{\partial\mathfrak{A}'}{\partial v} - \frac{\partial\mathfrak{A}''}{\partial u}\right),$$

$$\frac{\partial\mathfrak{A}'}{\partial\eta} - \frac{\partial\mathfrak{A}''}{\partial\xi} = \Sigma\frac{\partial}{\partial\eta}(pq)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(q^2)\mathfrak{A} + qW\left(\frac{\partial\mathfrak{A}}{\partial v} - \frac{\partial\mathfrak{A}'}{\partial u}\right) + q_1W\left(\frac{\partial\mathfrak{A}'}{\partial v} - \frac{\partial\mathfrak{A}''}{\partial u}\right).$$

And in virtue of (18), we have

$$\Sigma\frac{\partial}{\partial\eta}(p^2)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(pq)\mathfrak{A} = W\varphi(\mathfrak{A}),$$

$$\Sigma\frac{\partial}{\partial\eta}(pq)\mathfrak{A} - \Sigma\frac{\partial}{\partial\xi}(q^2)\mathfrak{A} = W\psi(\mathfrak{A}).$$

But we have as in **n°4.**

$$|\overline{\mathfrak{A}}\ \overline{D}_\xi\ \overline{D}| = -2W\varphi(\mathfrak{A}) + pW|\mathfrak{A}\ D_u\ D| + p_1W|\mathfrak{A}\ D_v\ D|,$$

$$|\overline{\mathfrak{A}}\ \overline{D}_\eta\ \overline{D}| = -2W\psi(\mathfrak{A}) + qW|\mathfrak{A}\ D_u\ D| + q_1W|\mathfrak{A}\ D_v\ D|.$$

Therefore, we have

$$\left.\begin{aligned} &\frac{\partial\overline{\mathfrak{A}}}{\partial\eta} - \frac{\partial\overline{\mathfrak{A}}'}{\partial\xi} + \frac{1}{2}|\overline{\mathfrak{A}}\ \overline{D}_\xi\ \overline{D}| \\ &\quad = pW\left\{\frac{\partial\mathfrak{A}}{\partial\eta} - \frac{\partial\mathfrak{A}'}{\partial\xi} + \frac{1}{2}|\mathfrak{A}\ D_u\ D|\right\} \\ &\quad + p_1W\left\{\frac{\partial\mathfrak{A}'}{\partial\eta} - \frac{\partial\mathfrak{A}''}{\partial\xi} + \frac{1}{2}|\mathfrak{A}\ D_v\ D|\right\}, \\ &\frac{\partial\overline{\mathfrak{A}}'}{\partial\eta} - \frac{\partial\overline{\mathfrak{A}}''}{\partial\xi} + \frac{1}{2}|\overline{\mathfrak{A}}\ \overline{D}_u\ \overline{D}| \end{aligned}\right\} \quad \cdots\cdots\cdots(34)$$

$$\left.\begin{aligned}
&=qW\left\{\frac{\partial\mathfrak{A}}{\partial\eta}-\frac{\partial\mathfrak{A}'}{\partial\xi}+\frac{1}{2}|\,\mathfrak{A}\,D_u D\,|\right\}\\
&+q_1W\left\{\frac{\partial\mathfrak{A}'}{\partial\eta}-\frac{\partial\mathfrak{A}''}{\partial\xi}+\frac{1}{2}|\,\mathfrak{A}\,D_v D\,|\right\}
\end{aligned}\right|$$

Similarly, we have

$$\left.\begin{aligned}
&\frac{\partial\overline{\mathfrak{L}}}{\partial\eta}-\frac{\partial\overline{\mathfrak{L}}'}{\partial\xi}+\frac{1}{2}|\,\overline{\mathfrak{L}}\;\overline{D}_\xi\;\overline{D}\,|\\
&\quad=pW\left\{\frac{\partial\mathfrak{L}}{\partial\eta}-\frac{\partial\mathfrak{L}'}{\partial\xi}+\frac{1}{2}|\,\mathfrak{L}\,D_u D\,|\right\}\\
&\quad+p_1W\left\{\frac{\partial\mathfrak{L}'}{\partial\eta}-\frac{\partial\mathfrak{L}''}{\partial\xi}+\frac{1}{2}|\,\mathfrak{L}\,D_v D\,|\right\}\\
&\frac{\partial\overline{\mathfrak{L}}'}{\partial\eta}-\frac{\partial\overline{\mathfrak{L}}''}{\partial\xi}+\frac{1}{2}|\,\overline{\mathfrak{L}}\;\overline{D}_\eta\;\overline{D}\,|\\
&\quad=qW\left\{\frac{\partial\mathfrak{L}}{\partial\eta}-\frac{\partial\mathfrak{L}'}{\partial\xi}+\frac{1}{2}|\,\mathfrak{L}\,D_u D\,|\right\}\\
&\quad+q_1W\left\{\frac{\partial\mathfrak{L}'}{\partial\eta}-\frac{\partial\mathfrak{L}''}{\partial\xi}+\frac{1}{2}|\,\mathfrak{L}\,D_v D\,|\right\}.
\end{aligned}\right\}\quad\cdots\cdots(35)$$

From (17) and (18) we have

$$\begin{aligned}
&\frac{\partial\overline{D}}{\partial\eta}-\frac{\partial\overline{D}'}{\partial\xi}+\frac{\partial\log\theta}{\partial\eta}\overline{D}-\frac{\partial\log\theta}{\partial\xi}\overline{D}'\\
&\quad=\varphi(D)+p\left\{\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}+\frac{\partial\log\theta}{\partial v}D-\frac{\partial\log\theta}{\partial u}D'\right\}\\
&\qquad+p_1\left\{\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}+\frac{\partial\log\theta}{\partial v}D'-\frac{\partial\log\theta}{\partial u}D''\right\},
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial\overline{D}'}{\partial\eta}-\frac{\partial\overline{D}''}{\partial\xi}+\frac{\partial\log\theta}{\partial\eta}\overline{D}'-\frac{\partial\log\theta}{\partial\xi}\overline{D}''\\
&\quad=\psi(D)+q\left\{\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}+\frac{\partial\log\theta}{\partial v}D-\frac{\partial\log\theta}{\partial u}D'\right\}\\
&\qquad+q_1\left\{\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}+\frac{\partial\log\theta}{\partial v}D'-\frac{\partial\log\theta}{\partial u}D''\right\}.
\end{aligned}$$

While we have as in **n°4.**

$$|\overline{D}_\xi\ \overline{D}_\eta\ \overline{D}|=2\frac{\partial\varphi(D)}{\partial\eta}-2\frac{\partial\psi(D)}{\partial\xi}+W|D_u D_v D|.$$

Therefore, if we put

$$\Theta=\frac{1}{2}\Big[\frac{\partial^2 D}{\partial v^2}-2\frac{\partial^2 D'}{\partial u\partial v}+\frac{\partial^2 D''}{\partial u^2}-\frac{1}{2}|D_u D_v D|$$

$$\frac{\partial}{\partial v}\Big(\frac{\partial\log\theta}{\partial v}D-\frac{\partial\log\theta}{\partial u}D'\Big)-\frac{\partial}{\partial u}\Big(\frac{\partial\log\theta}{\partial v}D'-\frac{\partial\log\theta}{\partial u}D''\Big)\Big],$$

we have

$$\overline{\Theta}=W\Theta. \quad \text{.....(36)}$$

Accordingly, Θ is a relative invariant and $\frac{\Theta}{\theta}$ is an absolute invariant.

From (33), (34), (35) and (36) we know that if we put

$$\mathfrak{F}^{(1)}=\frac{\partial}{\partial v}(2\mathfrak{A}-\mathfrak{L})-\frac{\partial}{\partial u}(2\mathfrak{A}'-\mathfrak{L}')$$

$$+\frac{1}{2}|2\mathfrak{A}-\mathfrak{L}\ D_u D|-|\mathfrak{L}\ \mathfrak{C}\ D|+\theta\frac{\partial}{\partial u}\Big(\frac{\Theta}{\theta}\Big),$$

$$\mathfrak{F}^{(2)}=\frac{\partial}{\partial v}(2\mathfrak{A}'-\mathfrak{L}')-\frac{\partial}{\partial u}(2\mathfrak{A}''-\mathfrak{L}'')$$

$$+\frac{1}{2}|2\mathfrak{A}-\mathfrak{L}\ D_v D|-|\mathfrak{L}\ \mathfrak{B}\ D|+\theta\frac{\partial}{\partial v}\Big(\frac{\Theta}{\theta}\Big),$$

we have

$$\left.\begin{aligned}\overline{\mathfrak{F}}^{(1)}&=pW\mathfrak{F}^{(1)}+p_1W\mathfrak{F}^{(2)},\\ \overline{\mathfrak{F}}^{(2)}&=qW\mathfrak{F}^{(1)}+qW\mathfrak{F}^{(2)}\end{aligned}\right\} \quad \text{.....(37)}$$

and we can easily see that

$$\left.\begin{aligned}\mathfrak{F}^{(1)}=&2\Big(\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}\Big)+\frac{\partial^2 C}{\partial v^2}-2\frac{\partial^2 C'}{\partial u\partial v}+\frac{\partial^2 C''}{\partial u^2}\\ &+\frac{\partial}{\partial v}|C\,D_u D|-\frac{\partial}{\partial u}|B-D_v C\,D|\\ &+|A\,D_u D|-|L\,C\,D|,\end{aligned}\right|$$

$$\left.\begin{aligned}\mathfrak{F}^{(2)} = 2\left(\frac{\partial A'}{\partial v} - \frac{\partial A''}{\partial u}\right) + \frac{\partial^2 B}{\partial v^2} - 2\frac{\partial^2 B'}{\partial u \partial v} + \frac{\partial^2 B''}{\partial u^2} \\ \frac{\partial}{\partial v}|C - D_u B D| - \frac{\partial}{\partial u}|B D_v D| \\ + |A D_v D| - |L B D|.\end{aligned}\right\} (38)$$

From (20), (25), (31) and (34) we know that if we put

$$\begin{aligned}\mathfrak{F}^{(3)} = \theta\frac{\partial}{\partial v}\left[\frac{1}{\theta}\left\{\frac{\partial \mathfrak{A}}{\partial v} - \frac{\partial \mathfrak{A}'}{\partial u} + \frac{1}{2}|\mathfrak{A} D_u D| - |\mathfrak{A}\,\mathfrak{C} D|\right\}\right] \\ - \theta\frac{\partial}{\partial u}\left[\frac{1}{\theta}\left\{\frac{\partial \mathfrak{A}'}{\partial v} - \frac{\partial \mathfrak{A}''}{\partial u} + \frac{1}{2}|\mathfrak{A} D_v D| - |\mathfrak{A}\,\mathfrak{B} D|\right\}\right] \\ - \mathfrak{L}\mathfrak{A} D|,\end{aligned}$$

we have

$$\overline{\mathfrak{F}}^{(3)} = W^2 \mathfrak{F}^{(3)},$$

and we can easily see that

$$\left.\begin{aligned}\mathfrak{F}^{(3)} = \frac{\partial^2 A}{\partial v^2} - 2\frac{\partial^2 A'}{\partial u \partial v} + \frac{\partial A''}{\partial u^2} - \frac{\partial}{\partial v}|A C - D_u D| \\ + \frac{\partial}{\partial u}|A B - D_v D| - |L A D| \\ - \frac{1}{2}\frac{\partial \log\theta}{\partial v}\mathfrak{F}^{(1)} + \frac{1}{2}\frac{\partial \log\theta}{\partial u}\mathfrak{F}^{(2)}.\end{aligned}\right\} \cdots (39)$$

We shall show, in the subsequent chapter, that $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ and $\mathfrak{F}^{(3)}$ vanish identically.

CHAPTER II.

FUNDAMENTAL EQUATIONS OF THE PROJECTIVE DIFFERENTIAL GEOMETRY OF NONDEVELOPABLE SURFACES.

1. Put

$$y = D\frac{\partial x}{\partial v} - D'\frac{\partial x}{\partial u},$$

$$\left.\begin{aligned} z &= D'\frac{\partial x}{\partial v} - D''\frac{\partial x}{\partial u}, \\ w &= \frac{1}{2}\left(\frac{\partial y}{\partial v} - \frac{\partial z}{\partial u}\right). \end{aligned}\right\} \quad \cdots\cdots(1)$$

Then we have

$$|\, x\, y\, z\, w\,| = (DD'' - D'^2)^2 = 1.$$

Therefore, the four points x, y, z, w are not coplanar.
From (1) we have

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= -D'y + Dz, \\ \frac{\partial x}{\partial v} &= -D''y + D'z. \end{aligned}\right\} \quad \cdots\cdots(2)$$

From the definition of the fundamental determinants we have

$$\left.\begin{aligned} &Ax - B\frac{\partial x}{\partial u} + C\frac{\partial x}{\partial v} - D'\frac{\partial^2 x}{\partial u^2} + D\frac{\partial^2 x}{\partial u \partial v} = 0, \\ &2A'x - 2B'\frac{\partial x}{\partial u} + 2C'\frac{\partial x}{\partial v} - D''\frac{\partial^2 x}{\partial u^2} + D\frac{\partial^2 x}{\partial v^2} = 0, \\ &A''x - B''\frac{\partial x}{\partial u} + C''\frac{\partial x}{\partial v} - D''\frac{\partial^2 x}{\partial u \partial v} + D'\frac{\partial^2 x}{\partial v^2} = 0. \end{aligned}\right\} \quad \cdots\cdots(3)$$

From (1), (2) and (3) we have

$$\left.\begin{aligned} \frac{\partial y}{\partial u} &= [D'(B - \frac{\partial D}{\partial v}) - D(C'' - \frac{\partial D''}{\partial u})]y \\ &\quad - [D'(C - \frac{\partial D}{\partial u}) - D(B - \frac{\partial D'}{\partial u})]z - Ax, \\ \frac{\partial y}{\partial v} &= w + [D''(B - \frac{\partial D}{\partial v}) - D'(C'' - \frac{\partial D''}{\partial u})]y \\ &\quad - [D'(B - \frac{\partial D}{\partial v}) - D(C'' - \frac{\partial D''}{\partial u})]z - A'x, \\ \frac{\partial z}{\partial u} &= -w + [D''(B - \frac{\partial D}{\partial v}) - D'(C'' - \frac{\partial D''}{\partial u})]y \end{aligned}\right\} \quad \cdots(2_2)$$

$$-[D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial u})]z-A'x,$$

$$\frac{\partial z}{\partial v}=[D''(C''-\frac{\partial D'}{\partial v})-D'(B''-\frac{\partial D''}{\partial v})]y$$

$$-[D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial u})]z-A''x.$$

From (2_2) by writing

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right),$$

we have

$$\left.\begin{aligned}
\frac{\partial w}{\partial u}&=[D''(A-L)-D'(A'-L')+D'(\theta-\Theta_{oo})]y\\
&\quad-[D'(A-L)-D(A'-L')+D(\theta-\Theta_{oo})]z\\
&\quad-[\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}-|\,A\,C-D_u\,D\,|\,]x,\\
\frac{\partial w}{\partial v}&=[D''(A'-L')-D'(A''-L'')+D''(\theta-\Theta_{oo})]y\\
&\quad-[D'(A'-L')-D(A''-L'')+D'(\theta-\Theta_{oo})]z\\
&\quad-[\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}-|\,A\,B-D_v\,D\,|\,]x.
\end{aligned}\right\}\ldots\ldots(2_3)$$

where

$$\Theta_{oo}=\frac{1}{2}\frac{\partial^2 D}{\partial v^2}-\frac{\partial^2 D'}{\partial u\partial v}+\frac{1}{2}\frac{\partial^2 D''}{\partial u^2}-\frac{1}{4}|\,D_u\,D_v\,D\,|\,.$$

The equations (2_1) (2_2) and (2_3) form a system of simultaneous linear partial differential equations.

The conditions of the complete integrability of this system which is obtained by writing

$$\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)$$

are

$$\left.\begin{aligned}
&2\left(\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}\right)+\frac{\partial^2 C}{\partial v^2}-2\frac{\partial^2 C'}{\partial u\partial v}+\frac{\partial^2 C''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|\,C\,D_u\,D\,|-\frac{\partial}{\partial u}|\,C\,B-D_v\,D\,|\\
&\quad+|\,A\,D_u\,D\,|-|\,L\,C\,D\,|=0,\\
&2\left(\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}\right)+\frac{\partial^2 B}{\partial v^2}-2\frac{\partial^2 B'}{\partial u\partial v}+\frac{\partial^2 B''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|\,C-D_u\,B\,D\,|-\frac{\partial}{\partial w}|\,B\,D_v\,D\,|\\
&\quad+|\,A\,D_v\,D\,|-|\,L\,C\,D\,|=0,\\
&\frac{\partial^2 A}{\partial v^2}-2\frac{\partial^2 A'}{\partial u\partial v}+\frac{\partial^2 A''}{\partial u^2}-\frac{\partial}{\partial v}|\,A\,C-D_u\,D\,|\\
&\quad+\frac{\partial}{\partial u}|\,A\,B-D_v\,D\,|-|\,L\,A\,D\,|=0.
\end{aligned}\right\}\quad\cdots\cdots(4)$$

From (38), (39) (**Chap. I**) and (4) we know that $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ and $\mathfrak{F}^{(3)}$ vanish identically.

Let (X, Y, Z, W) a pair of the solutions of the system formed by the equations (2_1), (2_2), (2_3) and consider the system formed by the first equation of (2_1), the first and the third of (2_2), and the first of (2_3). If we regard v as a parameter, the latter system is that of the normal linear ordinary differential equations of which the normal coordinates x_i and y_i, z_i, w_i $(i=1, 2, 3, 4)$ defined by the equation (1) are a fundamental system of the solutions.[1]

Therefore, we have

$$X=\sum_{i=1}^{4}C_i(v)x_i,$$

$$Y=\sum_{i=1}^{4}C_i(v)y_i,$$

$$Z=\sum_{i=1}^{4}C_i(v)z_i,$$

[1] Poussin, Cours d'Analyse Infinitesimal, Vol. II. p. 256.

$$W=\sum_{i=1}^{4}C_i(v)w_i$$

where $C_i(v)$ $(i=1, 2, 3, 4)$ are functions of v only.

Substituting these values of X, Y, Z, W in the seconde quations of (2_1) the second and the fourth of (2_2), and the second of (2_3), we have

$$\sum_{i=1}^{4}\frac{dC_i}{dv}x_i=0, \quad \sum_{i=1}^{4}\frac{dC_i}{dv}y_i=0,$$

$$\sum_{i=1}^{4}\frac{dC_i}{dv}z_i=0, \quad \sum_{i=1}^{4}\frac{dC_i}{dv}w_i=0.$$

Therefore $C_i(i=1, 2, 3, 4)$ are constants.

Therefore, we know that if X_i, Y_i, Z_i, W_i $(i=1, 2, 3, 4)$ be four systems of the solutions which satisfy the relations

$$|X\,Y\,Z\,W|\neq 0,$$

The surface described by the point X is a projective transform of the given surface, in another words, *auy surface of which the normal fundamental determinants are equal to those of the surface S is a projective transform of S.*

But by the transformation

$$\bar{x}=e^{\frac{i\pi}{4}}x,$$

the signs of all the normal fundamental determinants are changed. Therefore, we know that *any surface of which the normal fundamental determinants are equal, except for a sign, to those of the surface S is a projective transform of S.*

2. Next, let us suppose that we are given twelve function D, D', D''; B, B', B''; C, C', C''; A, A', A'' which are holomorph in the domain R and satisfy the equations

$$\left.\begin{aligned} D'^2-DD''&=1,\\ DB''-2D'B'+DB''&=0,\\ DC''-2D'C'+DC''&=0,\\ DA''-2D'A'+DA''&=0, \end{aligned}\right\}$$

$$\left.\begin{aligned}
&\frac{\partial D}{\partial v}-\frac{\partial D'}{\partial u}=2(B-C')\\
&\frac{\partial D'}{\partial v}-\frac{\partial D''}{\partial u}=2(B'-C'')\\
&2\left(\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}\right)+\frac{\partial^2 C}{\partial v^2}-2\frac{\partial^2 C'}{\partial u\partial v}+\frac{\partial^2 C''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|\,C\,D_u\,D\,|-\frac{\partial}{\partial u}|\,C\,B-D_v\,D\,|\\
&\quad+|\,A\,D_u\,D\,|-|\,L\,C\,D\,|=0,\\
&2\left(\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial u}\right)+\frac{\partial^2 B}{\partial v^2}-2\frac{\partial^2 B'}{\partial u\partial v}+\frac{\partial^2 B''}{\partial u^2}\\
&\quad+\frac{\partial}{\partial v}|\,C-D_u\,B\,D\,|-\frac{\partial}{\partial u}|\,B\,D_v\,D\,|\\
&\quad+|\,A\,D_v\,D\,|-|\,L\,C\,D\,|=0,\\
&\frac{\partial^2 A}{\partial v^2}-2\frac{\partial^2 A'}{\partial u\partial v}+\frac{\partial^2 A''}{\partial u^2}-\frac{\partial}{\partial v}|\,A\,C-D_u\,D\,|\\
&\quad+\frac{\partial}{\partial u}|\,A\,B-D_v\,D\,|-|\,L\,A\,D\,|=0.
\end{aligned}\right\}\ \cdots\cdots(5)$$

Then the system of the total differential equations

$$\left.\begin{aligned}
dx&=-(D'y-Dz)du-(D''y-D'z)dv,\\
dy&=[\{D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial u})\}y\\
&\quad-\{D'(C-\frac{\partial D}{\partial u})-D(B-\frac{\partial D'}{\partial u})\}z-A'x]\,du\\
&+[w+\{D''(B-\frac{\partial D}{\partial v})-D'(C-\frac{\partial D''}{\partial w})\}y\\
&\quad-\{D'(B-\frac{\partial D}{\partial v})-D(C''-\frac{\partial D''}{\partial w})\}z-A'x]dv,\\
dz&=[-w+\{D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial w})\}y
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&-\{D'(B-\frac{\partial D}{\partial v}(-D(C''-\frac{\partial D''}{\partial w})\}z-A'x]du\\
&+[\{D''(C''-\frac{\partial D'}{\partial v})-D'(B''-\frac{\partial D''}{\partial v})\}y\\
&-\{D''(B-\frac{\partial D}{\partial v})-D'(C''-\frac{\partial D''}{\partial w})z-A''x]dv,\\
dw=&[\{D''(A-L)-D'(A'-L')+D'(\theta-\Theta_{oo})\}y\\
&-\{D'(A-L)-D(A'-L')+D(\theta-\Theta_{oo})\}z\\
&-\{\frac{\partial A}{\partial v}-\frac{\partial A'}{\partial u}-|AC-D_uD|\}x]du\\
&+[\{D''(A'-L')-D'(A''-L'')+D''(\theta-\Theta_{oo})\}y\\
&-\{D'(A'-L')-D(A''-L'')+D'(\theta-\Theta_{oo})\}z\\
&-\{\frac{\partial A'}{\partial v}-\frac{\partial A''}{\partial w}-|AB-D_vD|\}x]dv
\end{aligned}\right\}\ldots(6)$$

are completely integrable.

Therefore there are one and only one system of the functions x, y, z w which satisfy (6), and are holomorph in the vicinity of x_o, y_o, z_o, w_o, and assume respectively the values x_o, y_o, z_o, w_o for $u=u_o$, $v=v_o$, (u_o, v_o) being a point in the domain R.[1]

Take constants $(\lambda_i, \mu_i, \nu_i, \sigma_i)$ $(i=1, 2, 3, 4)$ which satisfy

$$|\lambda\,\mu\,\nu\,\sigma| = 1.$$

Let (x_i, y_i, z_i, w_i) $(i=1, 2, 3, 4)$ be the integrals which assume respectively the values $(\lambda_i, \mu_i, \nu_i, \sigma_i)$ for $u=u_o$, $v=v_o$.

From (6) we have

$$\frac{\partial}{\partial u}|x\,y\,z\,w| = 0,$$

$$\frac{\partial}{\partial v}|x\,y\,z\,w| = 0.$$

Therefore $|x\,y\,z\,w|$ is a constant,

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1 Grousat, Leçons sur l'integration des équation aux dérivées partielles du premier ordre, p. 73.

But

$$|x\,y\,z\,w| = 1$$

for $u=u_0$, $v=v_0$. Accordingly this relation holds for all values of u, v.

From (6) we have

$$\left.\begin{aligned}
\frac{\partial^2 x}{\partial u^2} &= -Dw-(2C'-B)y+Cz+(AD'-A'D)x,\\
\frac{\partial^2 x}{\partial u\partial v} &= -D'w-C''y+Bz+\frac{1}{2}(AD''-A''D)x,\\
\frac{\partial^2 x}{\partial v^2} &= -D''w-B''y+(2B'-C'')z+(A'D''-A''D')x.
\end{aligned}\right\}\ldots(7)$$

From (6) and (7) we have

$$\begin{aligned}
&|x\ x_u\ x_v\ x_{uu}| = D,\\
\frac{1}{2}&|x\ x_u\ x_v\ x_{uv}| = D',\\
&|x\ x_u\ x_v\ x_{vv}| = D'',\\
&|x\ x_v\ x_{uu}\ x_{uv}| = B,\\
\frac{1}{2}&|x\ x_v\ x_{uu}\ x_{vv}| = B',\\
&|x\ x_v\ x_{uv}\ x_{vv}| = B'',\\
&|x\ x_u\ x_{uu}\ x_{uv}| = C,\\
\frac{1}{2}&|x\ x_u\ x_{uu}\ x_{vv}| = C',\\
&|x\ x_u\ x_{uv}\ x_{vv}| = C'',\\
&|x_u\ x_v\ x_{uu}\ x_{uv}| = A,\\
\frac{1}{2}&|x_u\ x_v\ x_{uu}\ x_{vv}| = A',\\
&|x_u\ x_v\ x_{uv}\ x_{vv}| = A''.
\end{aligned}$$

Therefore we know that *if we are given twelve function of* u, v D, D', D''; B, B', B''; C, C', C''; A, A', A'' *which satisfy the equa-*

tion (5), *we have nnique surface which has them as the normal fundamental determinants, except for a projective transformation.*

3. Let S and S_1 be two nondevelopable surfaces which are projectively deformable to each other.

Let us suppose that they are defined by the equations

$$\left.\begin{aligned} \mathfrak{x}_i &= \mathfrak{x}_i(u, v), \\ \mathfrak{X}_i &= \mathfrak{X}_i(u, v), \end{aligned}\right\} \qquad (i = 1, 2, 3, 4)$$

respectively and that parameters u, v are so chosen that the corresponding points on S and S_1 correspond to the same values of u, v. Denote by a, b, ... ; a_1, b_1 ..... the fundamental determinants of S and S_1 respectively.

Fubini[1] has proved that for S and S_1 to be projectively deformable to each other, it is necessary and sufficient that

$$\left.\begin{aligned} d &= \lambda d_1, \quad d' = \lambda d_1', \quad d'' = \lambda d_1'', \\ b &= \lambda[b_1 - 2\beta d_1' + \gamma d_1], \\ b' &= \lambda[b_1' - \beta d_1''], \\ b'' &= \lambda[b_1'' - \gamma d_1''], \\ c &= \lambda[c_1 - \beta d_1], \\ c' &= \lambda[c_1' - \gamma d_1], \\ c'' &= \lambda[c_1'' - 2\gamma d_1' + \beta d_1''] \end{aligned}\right\} \qquad \text{.....(8)}$$

where λ, β, γ are functions of u, v.

From (8) we have

$$\Delta = \varepsilon\lambda\Delta_1 \qquad (\varepsilon = +1, \text{ or } -1) \text{.....(9)}$$

From (18) and the equations

$$\frac{\partial d}{\partial v} - \frac{\partial d'}{\partial u} = 2(b - c'),$$

$$\frac{\partial d'}{\partial v} - \frac{\partial d''}{\partial u} = 2(b' - c''),$$

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1 Rendiconti del circolo mathematico di Palermo. **41**, p. 135 (1916).

$$\frac{\partial d_1'}{\partial v}-\frac{\partial d_1'}{\partial u}=2(b_1-c_1'),$$

$$\frac{\partial d_1'}{\partial v}-\frac{\partial d_1''}{\partial u}=2(b_1'-c_1''),$$

we have

$$\left(\beta-\frac{1}{4}\frac{\partial\log\lambda}{\partial u}\right)d_1'-\left(\gamma-\frac{1}{4}\frac{\partial\log\lambda}{\partial v}\right)d_1=0,$$

$$\left(\beta-\frac{1}{4}\frac{\partial\log\lambda}{\partial v}\right)d_1''-\left(\gamma-\frac{1}{4}\frac{\partial\log\lambda}{\partial v}\right)d_1'=0.$$

Therefore we have

$$\left.\begin{aligned}&\beta-\frac{\partial\log\lambda}{\partial u}=0.\\&\gamma-\frac{1}{4}\frac{\partial\log\lambda}{\partial v}=0.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(10)$$

Let $A, B, \ldots\ldots; A_1, B_1, \ldots\ldots$ be the normal fundamental determinants of S and S_1 respectively.

Then we have from (8), (9) and (10)

$$\left.\begin{aligned}&D_1=\varepsilon D,\quad D_1'=\varepsilon D',\quad D_1''=\varepsilon D'',\\&B_1=\varepsilon B,\quad B_1'=\varepsilon B',\quad B_1''=\varepsilon B'',\\&C_1=\varepsilon C,\quad C_1'=\varepsilon C',\quad C_1''=\varepsilon C''.\end{aligned}\right\}\quad\ldots\ldots(11)$$

where $\varepsilon=+1$, or -1.

Therefore we know that for the two surfaces S and S_1 to be projectively deformable to each other, it is necessary and sufficient that we can choose the parameters u, v so that corresponding points on S and S_1 correspond to the same values of u, v and satisfy equation (11).

(*To be Continued.*)

On the Curvature of a Curve in n Dimensional Non-Euclidean Space.

By

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1. The curvature of a curve in n dimensional Non-Euclidean space has already been discussed by many professors—G. Kowalewski,[1] E. Stransky[2] and T. Nishiuchi[3] from different points of view. And also Prof. T. Kubota[4] has treated the problem following the method given by E. Rath[5] in Euclidean space. In this paper, I will give the generalization of Prof. Nishiuchi's discussion according to the idea of Meyer[6] which he applied in the generalization of Serret-Frenet's formula in the Euclidean case.

In n Dimentional Space.

2. Let x_o, x_1, x_2, ..., x_n be thc Weirstrass coordinates of point in n dimensional space whose measure of curvature is $\frac{1}{k^2}$ and

$$(xx)=x_o^2+x_1^2+x_2^2+\ldots\ldots+x_n^2=0$$

be the equation to the absolute.

Suppose

$$x_o=\varphi_o(t),\ x_1=\varphi_1(t),\ x_2=\varphi_2(t),\ \ldots\ldots,\ x_n=\varphi_n(t)$$

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1 Wiener Berichte, **120**, (1911).
2 Wiener Berichte, **121**, (1912).
3 Mem. Coll. Sci., Kyoto, **5**. No. 3. (1921).
4 Kubota, in Sendai, Tôhoku Math. J., **21**. No. 3, 4. (1922).
5,6 Jahresber. D. M. ver., **19**, (1910).

represent the coordinates of a point on a curve Γ, where t is a parameter and

$$(\varphi_0(t))^2+(\varphi_1(t))^2+(\varphi_2(t))^2+\cdots\cdots+(\varphi_n(t))^2=k^2.$$

Next, we consider any other curve Γ' defined by the equations

$$a_i=\psi_i(t)=x_i\sin\frac{r}{k}+y_i\cos\frac{r}{k}$$

where r is a function of t and

$$(xy)=0,\ (aa)=k^2$$

$$i=0,\ 1,\ 2,\ \cdots\cdots,\ n.$$

Take a point $P_0[x(t)]$ on the curve Γ and take $r+1$ points $P_0'[a(t)]$, $P_1'[a(t+\Delta t)]$, $P_2'[a(t+2\Delta t)]$, $\cdots\cdots$, $P_r'[a(t+r\Delta t)]$ on the curve Γ'. Then the coordinates of a point ${}_rQ({}_r\eta)$ which is on the plane manifold passing through $r+1$ points $(P_0,\ P_0',\ P_1',\ \cdots\cdots,\ P_{r-1}')$ and orthogonol to r points $(P_0,\ P_0',\ P_1',\ \cdots\cdots,\ P_{r-2}')$ will be

$$(1)\quad {}_r\eta_i=\frac{k\begin{vmatrix}(xx) & (xa) & \cdots\cdots(xa^{r-2}) & x_i\\ (ax) & (aa) & \cdots\cdots(aa^{r-2}) & a_i\\ \vdots & \vdots & & \vdots\\ (a^{r-1}x) & (a^{r-1}a) & \cdots\cdots(a^{r-1}a^{r-2}) & a_i^{r-1}\end{vmatrix}}{\sqrt{\begin{vmatrix}(xx) & (xa) & \cdots(xa^{r-2})\\ (ax) & (aa) & \cdots(aa^{r-2})\\ \vdots & \vdots & \\ (a^{r-2}x) & (a^{r-2}a) & \cdots(a^{r-2}a^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(xx) & (xa) & \cdots(xa^{r-1})\\ (ax) & (aa) & \cdots(aa^{r-1})\\ \vdots & \vdots & \\ (a^{r-1}x) & (a^{r-1}a) & \cdots(a^{r-1}a^{r-1})\end{vmatrix}}},$$

$$(i=0,\ 1,\ 2,\ \cdots\cdots,\ n)$$

where

$$a^i=\frac{d^ia}{dt^i}.$$

And it is evident that

$$({}_r\eta,\eta)=0,\ (r\neq p),$$

$$({}_r\eta,\eta)=k^2,\ (r=p).$$

Next, let ${}_r\dot{Q}({}_r\dot{\eta})$ be a point on the plane manifold passing through $r+1$ points $(P_0,\ P_0',\ P_1',\ \cdots P_{r-2}',\ P_r')$ and orthogonal to r points $(P_0,\ P_0',\ P_1',\ \cdots,\ P_{r-2}')$, then

$$ {}_r\dot{\eta}_i = \left[\begin{vmatrix} (xx) & (x\alpha) & \dots & (x\alpha^{r-2}) & x_i \\ (\alpha x) & (\alpha\alpha) & \dots & (\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & & \vdots & \vdots \\ (\alpha^{r-1}x) & (\alpha^{r-1}\alpha) & \dots & (\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1} \end{vmatrix} + \begin{vmatrix} (xx) & (x\alpha) & \dots & (x\alpha^{r-2}) & x_i \\ \vdots & \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & (\alpha^{r-2}\alpha) & \dots & (\alpha^{r-2}\alpha^{r-2}) & \alpha_i^{r-2} \\ (\alpha^r x) & (\alpha^r\alpha) & \dots & (\alpha^r\alpha^{r-2}) & \alpha_i^r \end{vmatrix} \Delta t \right] \times $$

(2)

$$ \times \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) \\ (\alpha x) & \dots & (\alpha\alpha^{r-2}) \\ \vdots & & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) \end{vmatrix} \left[\begin{vmatrix} (xx) & \dots & (x\alpha^{r-1}) \\ (\alpha x) & \dots & (\alpha\alpha^{r-1}) \\ \vdots & & \vdots \\ (\alpha^{r-1}x) & \dots & (\alpha^{r-1}\alpha^{r-1}) \end{vmatrix} \right]^{\frac{1}{2}} $$

$$ + 2 \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^r) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-1}x) & \dots & (\alpha^{r-1}\alpha^{r-2}) & (\alpha^{r-1}\alpha^r) \end{vmatrix} \Delta t + \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^r) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^r) \\ (\alpha^r x) & \dots & (\alpha^r\alpha^{r-2}) & (\alpha^r\alpha^r) \end{vmatrix} \overline{\Delta t^2} \Bigg]^{-\frac{1}{2}}. $$

The limiting value of $\dfrac{\overline{{}_rQ\,{}_r\dot{Q}}}{\text{arc } P_0P_1} \times \dfrac{1}{k}$ *when* $P_1[x(t+\Delta t)]$ *approaches to* P_0 *shall be called the rth curvature of the curve* Γ *at the point* P_0 *with respect to the curve* Γ', in analogy to Prof. Nishiuchi's definition.[1]

But

$$ \sin^2 \frac{\overline{{}_rQ\,{}_r\dot{Q}}}{k} = \frac{\left\| \begin{matrix} {}_r\eta_0 & {}_r\eta_1 & \dots\dots & {}_r\eta_n \\ {}_r\dot{\eta}_0 & {}_r\dot{\eta}_1 & \dots\dots & {}_r\dot{\eta}_n \end{matrix} \right\|^2}{k^4} $$

$$ (3) \quad = \frac{\left\| \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^{r-1}) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^{r-1}) \\ (\alpha^{r-1}x) & \dots & (\alpha^{r-1}\alpha^{r-2}) & (\alpha^{r-1}\alpha^{r-1}) \end{vmatrix} \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^r) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^r) \\ (\alpha^{r-1}x) & \dots & (\alpha^{r-1}\alpha^{r-2}) & (\alpha^{r-1}\alpha^r) \end{vmatrix} \right\| \overline{\Delta t^2}}{\left\| \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^{r-1}) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^{r-1}) \\ (\alpha^{r-1}x) & \dots & (\alpha^{r-1}\alpha^{r-2}) & (\alpha^{r-1}\alpha^{r-1}) \end{vmatrix} \begin{vmatrix} (xx) & \dots & (x\alpha^{r-2}) & (x\alpha^r) \\ \vdots & & \vdots & \vdots \\ (\alpha^{r-2}x) & \dots & (\alpha^{r-2}\alpha^{r-2}) & (\alpha^{r-2}\alpha^r) \\ (\alpha^r x) & \dots & (\alpha^r\alpha^{r-2}) & (\alpha^r\alpha^r) \end{vmatrix} \right\|^2} $$

$$ + \theta\overline{\Delta t^2} $$

[1] Loc. cit.

where

$$\lim_{\Delta t \to 0.} \theta = 0.$$

Applying the Sylvester theorem, we have

$$\sin^2 \frac{\overline{{}_rQ_r\dot{Q}}}{k} = \frac{\triangle_{r-1}\triangle_{r+1}}{\triangle_r^2}\overline{\Delta t^2} + \theta\overline{\Delta t^2} \tag{4}$$

where

$$\triangle_i = \begin{vmatrix} (xx) & (x\alpha) & \dots (x\alpha^{i-1}) \\ \vdots & \vdots & \vdots \\ (\alpha^{i-1}x) & (\alpha^{i-1}\alpha) & \dots (\alpha^{i-1}\alpha^{i-1}) \end{vmatrix} = \begin{Vmatrix} x_0 & x_1 & \dots x_n \\ \alpha_0 & \alpha_1 & \dots \alpha_n \\ \alpha_0' & \alpha_1' & \dots \alpha_n' \\ \dots\dots\dots\dots\dots \\ \alpha_0^{i-1} & \alpha_1^{i-1} & \dots \alpha_n^{i-1} \end{Vmatrix}^2. \tag{5}$$

On the other hand

$$\sin \frac{\overline{P_oP_1}}{k} = \sqrt{\frac{\begin{vmatrix} (xx) & (xx') \\ (x'x) & (x'x') \end{vmatrix}}{(xx)^2}}\,\Delta t + W\Delta t$$

$$= \frac{\sqrt{\triangle}}{k^2}\Delta t + W\Delta t$$

where

$$\lim_{\Delta t \to 0} W = 0.$$

So the r^{th} curvature $\frac{1}{R_r}$ of the curve Γ at the point P_o with respect to the curve Γ' will be

$$\frac{1}{R_r} = \lim_{\Delta t \to 0} \frac{\overline{{}_rQ_r\dot{Q}}}{k \operatorname{arc} P_oP_1} = \lim_{\Delta t \to 0} \frac{k \sin \frac{\overline{{}_rQ_r\dot{Q}}}{k}}{k \sin \frac{\overline{P_oP_1}}{k}} \times \frac{1}{k}$$

$$= k\frac{\sqrt{\triangle_{r-1}\triangle_{r+1}}}{\sqrt{\triangle}\,\triangle_r} \tag{6}$$

$$(r = 1, 2, \dots\dots, n)$$

$$\frac{1}{R_0}=\frac{1}{k}.$$

Also, if ${}_r\theta$ be the angle between two plane manifolds whose absolute poles are ${}_rQ$ and ${}_r\dot{Q}$ respectively, then we have

$$\frac{1}{R_r}=\lim_{\Delta t\to 0}\frac{{}_r\theta}{\operatorname{arc} P_0P_1}.$$

From this equation we can also define $\frac{1}{R_r}$, but I think the former definition is natural.

3. Differentiating ${}_r\eta_i$ with respect to t, we have

$$\frac{d_r\eta_i}{dt}=\frac{k}{\triangle_{r-1}^{\frac{3}{2}}\triangle_r^{\frac{3}{2}}}\left[\triangle_{r-1}\left\{\begin{vmatrix}(xx) & \dots(x\alpha^{r-1})\\ \vdots & \\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-1})\end{vmatrix}\cdot\begin{vmatrix}(xx) & \dots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2}) & & \alpha_i^{r-2}\\ (\alpha^rx) & \dots(\alpha^r\alpha^{r-2}) & \alpha_i^r\end{vmatrix}\right.\right.$$

$$\left.-\begin{vmatrix}(xx) & \dots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2}) & & \alpha_i^{r-2}\\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \dots(x\alpha^{r-2}) & (x\alpha^r)\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2}) & & (\alpha^{r-2}\alpha^r)\\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-2}) & & (\alpha^{r-1}\alpha^r)\end{vmatrix}\right\}$$

$$-\triangle_r\left\{\begin{vmatrix}(xx) & \dots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2}) & & \alpha_i^{r-2}\\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \dots(x\alpha^{r-3}) & (x\alpha^{r-1})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-3}x)\dots(\alpha^{r-3}\alpha^{r-3}) & & (\alpha^{r-3}\alpha^{r-1})\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-3}) & & (\alpha^{r-2}\alpha^{r-1})\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx) & \dots(x\alpha^{r-2})\\ \vdots & \vdots\\ (\alpha^{r-3}x)\dots(\alpha^{r-3}\alpha^{r-2})\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\cdot\begin{vmatrix}(xx) & \dots(x\alpha^{r-3}) & (x\alpha^{r-1}) & x_i\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-3}) & & (\alpha^{r-2}\alpha^{r-1}) & \alpha_i^{r-2}\\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-3}) & & (\alpha^{r-1}\alpha^{r-1}) & \alpha_i^{r-1}\end{vmatrix}\right\}$$

$$+\triangle_r\left\{\begin{vmatrix}(x'x) & (x'a) & \dots(x'\alpha^{r-2}) & x_i'\\ (ax) & (aa) & \dots(a\alpha^{r-2}) & a_i\\ \vdots & \vdots & & \vdots\\ (\alpha^{r-1}x) & (\alpha^{r-1}a)\dots(\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx) & \dots(x\alpha^{r-2})\\ \vdots & \vdots\\ (\alpha^{r-2}x)\dots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix}(xx) & \dots(x\alpha^{r-2}) & x_i\\ \vdots & \vdots & \vdots\\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{r-1}\end{vmatrix}\cdot\begin{vmatrix}(xx') & (xa) & \dots(x\alpha^{r-2})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x')(\alpha^{r-2}a)\dots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\right\}$$

$$-\triangle_{r-1}\left\{\begin{vmatrix} (xx) & \dots (x\alpha^{r-2}) & x_i \\ \vdots & \vdots & \vdots \\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1} \end{vmatrix} \cdot \begin{vmatrix} (xx') & (x\alpha) & \dots (x\alpha^{r-1}) \\ \vdots & \vdots & \vdots \\ (\alpha^{r-1}x') & (\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-1}) \end{vmatrix}\right.$$

$$\left. -\begin{vmatrix} (xx') & (x\alpha) & \dots (x\alpha^{r-2}) & x_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}x') & (\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1} \end{vmatrix} \cdot \begin{vmatrix} (xx) & \dots (x\alpha^{r-1}) \\ \vdots & \vdots \\ (\alpha^{r-1}x)\dots(\alpha^{r-1}\alpha^{r-1}) \end{vmatrix}\right\}.$$

By the Sylvester theorem, we have

$$\frac{d_r\eta_i}{dt} = \frac{k}{\triangle_{r-1}^{\frac{3}{2}}\triangle_r^{\frac{3}{2}}}\left[(\triangle_{r-1}^{2}N_{r+1} - \triangle_r^{2}N_{r-1})\right]$$

$$+\triangle_r \begin{vmatrix} (\alpha x) & (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) \\ (\alpha' x) & (\alpha'\alpha) & \dots (\alpha'\alpha^{r-2}) \\ \vdots & \vdots & \vdots \\ (\alpha^{r-1}x)(\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-2}) \end{vmatrix} \cdot \begin{vmatrix} (x'x) & (x'\alpha) & \dots (x'\alpha^{r-2}) & x'_i \\ (xx) & (x\alpha) & \dots (x\alpha^{r-2}) & x_i \\ (\alpha x) & (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-2}x) & (\alpha^{r-2}\alpha)\dots(\alpha^{r-2}\alpha^{r-2}) & \alpha^{r-2} \end{vmatrix}$$

(7)

$$-\triangle_{r-1}\begin{vmatrix} (xx') & (xx) & (x\alpha) & \dots (x\alpha^{r-2}) \\ (\alpha x') & (\alpha x) & (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}x') & (\alpha^{r-1}x) & (\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-2}) \end{vmatrix} \cdot \begin{vmatrix} (x\alpha) & (x\alpha') & \dots (x\alpha^{r-1}) & x_i \\ (\alpha\alpha) & (\alpha\alpha') & \dots (\alpha\alpha^{r-1}) & \alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}\alpha)(\alpha^{r-1}\alpha')\dots(\alpha^{r-1}\alpha^{r-1}) & \alpha_i^{r-1} \end{vmatrix}$$

where

$$N_r = \begin{vmatrix} (xx) & (x\alpha) & \dots (x\alpha^{r-2}) & x_i \\ (\alpha x) & (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}x) & (\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-2}) & \alpha_i^{r-1} \end{vmatrix}$$

and

$$i = 0,\ 1,\ 2,\ \dots,\ n$$

$$r = 2,\ 3,\ \dots,\ n.$$

Hence, we have

$$\frac{d_r\eta_i}{ds} = \frac{{}_{r+1}\eta_i}{R_r} - \frac{{}_{r-1}\eta_i}{R_{r-1}} + \frac{r\xi_i}{P_r} - \frac{r\zeta_i}{Q_r},$$

$$(r=2, 3, \dots, n)\ (i=0, 1, 2, \dots, n).$$

$$(8)\qquad \frac{d_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},\quad \frac{d_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1}-\frac{\zeta_i}{Q_1},$$

$$\frac{d_{n+1}\eta_i}{ds}=-\frac{{}_n\eta_i}{R_n}+\frac{{}_{n+1}\xi_i}{P_{n+1}}-\frac{{}_{n+1}\zeta_i}{Q_{n+1}}$$

$$(i=0, 1, 2, \dots, n)$$

These are the generalized Serret-Frenet formula, where

$${}_r\xi_i=\frac{k\begin{vmatrix}(x'x) & (x'\alpha) & \dots(x'\alpha^{r-2}) & x'_i\\ (xx) & (x\alpha) & \dots(x\alpha^{r-2}) & x_i\\ (\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-2}) & \alpha_i\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & (\alpha^{r-2}\alpha)\dots(\alpha^{r-2}\alpha^{r-2}) & & \alpha_i^{r-2}\end{vmatrix}}{\sqrt{\begin{vmatrix}(xx) & (x\alpha) & \dots(x\alpha^{r-2})\\ (\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-2})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-2}x) & (\alpha^{r-2}\alpha)\dots(\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\begin{vmatrix}(x'x') & (x'x) & (x'\alpha) & \dots(x'\alpha^{r-2})\\ (xx') & (xx) & (x\alpha) & \dots(x\alpha^{r-2})\\ (\alpha x') & (\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-2})\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-2}x') & (\alpha^{r-2}x) & (\alpha^{r-2}\alpha)\dots & (\alpha^{r-2}\alpha^{r-2})\end{vmatrix}}},$$

$$(9)\qquad {}_r\zeta_i=\frac{k\begin{vmatrix}(x\alpha) & (x\alpha') & \dots(x\alpha^{r-1}) & x_i\\ (\alpha\alpha) & (\alpha\alpha') & \dots(\alpha\alpha^{r-1}) & \alpha_i\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-1}\alpha) & (\alpha^{r-1}\alpha')\dots(\alpha^{r-1}\alpha^{r-1}) & & \alpha_i^{r-1}\end{vmatrix}}{\sqrt{\begin{vmatrix}(\alpha\alpha) & \dots(\alpha\alpha^{r-1})\\ \vdots & \vdots\\ (\alpha^{r-1}\alpha)\dots(\alpha^{r-1}\alpha^{r-1})\end{vmatrix}\begin{vmatrix}(xx) & (x\alpha) & \dots(x\alpha^{r-1})\\ (\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-1})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-1}x) & (\alpha^{r-1}\alpha)\dots & (\alpha^{r-1}\alpha^{r-1})\end{vmatrix}}},$$

$$\frac{1}{P_r}=\frac{k}{\triangle}\frac{\sqrt{\begin{vmatrix}(x'x') & (x'x) & (x'\alpha) & \dots(x'\alpha^{r-2})\\ (xx') & (xx) & (x\alpha) & \dots(x\alpha^{r-2})\\ (\alpha x') & (\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-2})\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha^{r-2}x') & (\alpha^{r-2}x) & (\alpha^{r-2}\alpha)\dots & (\alpha^{r-2}\alpha^{r-2})\end{vmatrix}\begin{vmatrix}(\alpha x) & (\alpha\alpha) & \dots(\alpha\alpha^{r-2})\\ (\alpha' x) & (\alpha'\alpha) & \dots(\alpha'\alpha^{r-2})\\ \vdots & \vdots & \vdots\\ (\alpha^{r-1}x) & (\alpha^{r-1}\alpha)\dots & (\alpha^{r-1}\alpha^{r-2})\end{vmatrix}}}{\sqrt{\triangle_r}\qquad\qquad\triangle_{r-1}}$$

$$\frac{1}{Q_r}=\frac{k}{\Delta}\frac{\sqrt{\begin{vmatrix}(aa) & \dots & (aa^{r-1})\\ \vdots & & \vdots\\ (aa^{r-1}) & \dots & (a^{r-1}a^{r-1})\end{vmatrix}}}{\sqrt{\Delta_{r-1}}}\frac{\begin{vmatrix}(xx') & (xx) & (xa) & \dots & (xa^{r-2})\\ (ax') & (ax) & (aa) & \dots & (aa^{r-2})\\ \vdots & \vdots & \vdots & & \vdots\\ (a^{r-1}x') & (a^{r-1}x) & (a^{r-1}a) & \dots & (a^{r-1}a^{r-2})\end{vmatrix}}{\Delta_r}.$$

It is easily seen that $({}_r\xi)$ will be the coordinates of a point ${}_rX$ which is on the plane manifold passing through $r+1$ points $(P_0, P_1, P_0', P_1', \dots, P'_{r-2})$ and orthogonal to r points $(P_0, P_0', P_1', \dots, P_{r-2}')$. And $({}_r\zeta)$ be that of a point ${}_rZ$ which is on the plane manifold passing through $r+1$ points $(P_0, P_0', P_1', \dots, P_{r-1}')$ and orthogonal to r points $(P_0', P_1', \dots, P'_{r-1})$.

Next, let Δr be the difference of the distance between the points $P_0[x(t)]$, ${}_rQ'[{}_r\eta(t+\Delta t)]$ and $P_0[x(t)]$, ${}_rQ[{}_r\eta(t)]$, then

$$\frac{1}{Q_r}=\frac{1}{\cos\frac{\overline{P_0\,{}_rZ}}{k}}\left[\lim_{\Delta t\to 0}\frac{\Delta r}{\operatorname{arc} P_0P_1}\right],$$

$$\frac{1}{P_r}=\frac{\cos\frac{\overline{{}_rQ_rZ}}{k}}{\cos\frac{\overline{{}_rQ_rX}}{k}}\frac{1}{Q_r}.$$

We shall call $\frac{1}{P_r}\left(\frac{1}{Q_r}\right)$ the *r^{th} left (right) curvature of the curve* Γ at the point $P_0(x)$ with respect to the curve Γ'.

4. Specially, if Γ' be a curve traced by a point on the tangent to the curve Γ at the point P_0, then we have

$$a_i=x_i\sin\frac{r}{k}+\rho k x_i'\cos\frac{r}{k},$$

where

$$\rho=\frac{dt}{ds},$$

$$(i=0, 1, 2, 3, \dots, n).$$

And we have

$$(10)\qquad {}_r\eta_i = \frac{k\begin{vmatrix} (xx) & (xx') & \dots & (xx^{r-1}) & x_i \\ (x'x) & (x'x') & \dots & (x'x^{r-1}) & x'_i \\ \vdots & \vdots & & \vdots & \vdots \\ (x^r x) & (x^r x') & \dots & (x^r x^{r-1}) & x^r_i \end{vmatrix}}{\sqrt{\begin{vmatrix} (xx) & \dots & (xx^{r-1}) \\ \vdots & & \vdots \\ (x^{r-1}x) & \dots & (x^{r-1}x^{r-1}) \end{vmatrix} \begin{vmatrix} (xx) & \dots & (xx^r) \\ \vdots & & \vdots \\ (x^r x) & \dots & (x^r x^r) \end{vmatrix}}},$$

$$(11)\qquad \frac{1}{R_r^2} = k^2 \frac{\triangle_{r-1}\triangle_{r+1}}{\triangle_r^2 \triangle_1},$$

where

$$\triangle_i = \begin{vmatrix} (xx) & \dots & (xx^i) \\ \vdots & & \vdots \\ (x^i x) & \dots & (x^i x^i) \end{vmatrix}.$$

And the Serret-Frenet formula will be

$$\frac{d_r \eta_i}{ds} = \frac{{}_{r+1}\eta_i}{R_r} - \frac{{}_{r-1}\eta_i}{R_{r-1}}, \quad (r = 1, 2, 3, \dots, n-1)$$

$$(12)\qquad \frac{d_0 \eta_i}{ds} = \frac{{}_1\eta_i}{k}, \quad \frac{d_n \eta_i}{ds} = -\frac{{}_{n-1}\eta_i}{R_{n-1}},$$

$$(i = 0, 1, 2, \dots, n).$$

There are the formula given by Prof. Nishiuchi.[1]

In Euclidean Space.

5. If we put

$$x_0 = k\dot{x}_0, \quad x_1 = \dot{x}_1, \quad x_2 = \dot{x}_2, \quad \dots, \quad x_n = \dot{x}_n,$$

$$y_0 = k\dot{y}_0, \quad y_1 = \dot{y}_1, \quad y_2 = \dot{y}_2, \quad \dots, \quad y_n = \dot{y}_n$$

and

$$\frac{\dot{x}_i}{\dot{x}_0} = x_i,$$

$$\frac{\dot{y}_i}{\dot{y}_0} = y_i, \quad (i = 1, 2, \dots, n)$$

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1 loc. cit.

and take the limiting value when $\frac{1}{k^2}$ tends to zero, we have

$$\frac{1}{R_r^2}=\frac{D_{r-1}D_{r+1}}{\triangle D_r^2}, \tag{13}$$

where

$$D_i=\left\| \begin{matrix} y_1-x_1 & y_2-x_2 & \cdots\cdots & y_n-x_n \\ \bar{a}_1' & \bar{a}_2' & \cdots\cdots & \bar{a}_n' \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \bar{a}_1^{i-1} & \bar{a}_2^{i-1} & & \bar{a}_n^{i-1} \end{matrix} \right\|^2$$

$$\bar{a}_i=x_i+r(y_i-x_i)$$

$$\triangle=(x'x').$$

6. If we consider the case when r tends to infinity in the formula above, then

$$\frac{1}{R_r^2}=\frac{D_{r-1}D_{r+1}}{\triangle D_r^2} \tag{14}$$

where

$$D_i'=\left\| \begin{matrix} a_1 & a_2 & \cdots a_n \\ a_1' & a_2' & \cdots a_n' \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_1^{i-1} & a_2^{i-1} & \cdots a_n^{i-1} \end{matrix} \right\|^2 \tag{14$_1$}$$

$$a_i=y_i-x_i.$$

And if we take

$$\lim_{k\to\infty}\frac{k_r\eta_i}{{}_r\eta_0}={}_ra_i,$$

$$({}_ra_r\alpha)=1,$$

then we have

$$(15)\qquad {}_r\alpha_1 = \frac{\begin{vmatrix} (\alpha\alpha) & (\alpha\alpha') & \dots (\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^{r-1}\alpha) & (\alpha^{r-1}\alpha') \dots (\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{r-1} \end{vmatrix}}{\sqrt{\begin{vmatrix} (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) \\ \vdots & \vdots \\ (\alpha^{r-2}\alpha) \dots (\alpha^{r-2}\alpha^{r-2}) \end{vmatrix} \cdot \begin{vmatrix} (\alpha\alpha) & \dots (\alpha\alpha^{r-1}) \\ \vdots & \vdots \\ (\alpha^{r-1}\alpha) \dots (\alpha^{r-1}\alpha^{r-1}) \end{vmatrix}}}$$

$$(r = 1,\ 2,\ \dots,\ n+1).$$

$${}_1\alpha_i = \alpha_i,$$

$$({}_r\alpha_p\alpha) = 0,\quad (\alpha \neq p)$$

$$({}_r\alpha_p\alpha) = 1 \quad (\alpha = p).$$

Now, instead of the curve Γ', we consider a straight line g through the point P_0 on the curve Γ whose direction cosines are $({}_1\alpha)$. If we take n consecutive lines g', g'', ..., g^n through the points P_1, P_2, ..., P_n respectively and let the lines p_1, p_2, ..., p_n be parallel to the lines g', g'' ....., g^n through the point P_0, then the equation (15) gives the direction cosines of the line $({}_r\alpha)$ which is in the plane manifold possing through the lines g, p_1, p_2, ..., p_{r-1} and orthogonal to the one passing through the lines g, p_1, p_2, ..., p_{r-2}.

And $\frac{1}{R_r}$ which is given by the equation (14) shall be called the *r^{th} curvature of the curve* Γ at the point P_0 with respect to the line g.

Let ${}_r\theta$ be the angle between the line $({}_r\alpha)$ and the line which is in the plane manifold passing g, p_1, p_2, ..., p_{r-2}, p_r and orthogonal to the one passing through g, p_1, p_2, ..., p_{r-2}, then it is evident that

$$\lim_{\Delta t \to 0} \frac{{}_r\theta}{\text{arc } P_0P_1} = \frac{1}{R_r}.$$

Differentiating ${}_r\alpha_i$ with respect to t, we have.

$$(16)\qquad \frac{d_r\alpha_i}{dt} = \frac{1}{D_{r-1}^{\frac{3}{2}} D_{r+1}^{\frac{3}{2}}} \{D_{r-1}^2 N_{r+1} - D_r^2 N_{r-1}\}$$

where

$$N_r = \begin{vmatrix} (\alpha\alpha) & \dots (\alpha\alpha^{r-2}) & \alpha_i \\ \vdots & \vdots & \vdots \\ (\alpha^{r-1}\alpha) \dots (\alpha^{r-1}\alpha^{r-2}) & & \alpha_i^{i-1} \end{vmatrix}.$$

Hence, we have

$$\frac{d_r a_i}{ds} = \frac{{}_{r+1}a_i}{R_r} - \frac{{}_{r-1}a_i}{R_{r-1}}, \quad (r = 2, 3, \ldots\ldots, n).$$

(17)
$$\frac{d_1 a_i}{ds} = \frac{{}_2 a_i}{R_1},$$

$$\frac{d_{n+1} a_i}{ds} = -\frac{{}_n a_i}{R_n}$$

$$(i = 1, 2, \ldots, n).$$

These are generalized Brunel[1] and Landsbergs[1] formula.

7. From the equations (17), we have

$$\left(\frac{d_r a}{ds}\ \frac{d_r a}{ds}\right) = \frac{1}{R_r^2} + \frac{1}{R_{r-1}^2} \equiv \frac{1}{r_r^2} (\equiv K_r^2),$$

$$\left(\frac{d_1 a}{ds}\ \frac{d_1 a}{ds}\right) = \frac{1}{R_1^2} \equiv \frac{1}{r_1^2} (\equiv K_1^2),$$

$$\left(\frac{d_{n+1} a}{ds}\ \frac{d_{n+1} a}{ds}\right) = \frac{1}{R_n^2} \equiv \frac{1}{r_{n+1}^2} (\equiv K_{n+1}^2).$$

So we get

$$\frac{1}{R_2^2} = \frac{1}{r_2^2} - \frac{1}{r_1^2},$$

$$\frac{1}{R_3^2} = \frac{1}{r_3^2} - \frac{1}{R_2^2} = \frac{1}{r_3^2} - \frac{1}{r_2^2} + \frac{1}{r_1^2}$$

(18) .....

$$\frac{1}{R_i^2} = \frac{1}{r_i^2} - \frac{1}{R_{i-1}^2} = \frac{1}{r_i^2} - \frac{1}{r_{i-1}^2} + \frac{1}{r_{i-2}^2} \pm \ldots + (-1)^{i-1} \frac{1}{r_1^2},$$

.....

$$\frac{1}{R_n^2} = \frac{1}{r_n^2} - \frac{1}{R_{n-1}^2} = \frac{1}{r_n^2} - \frac{1}{r_{n-1}^2} + \ldots\ldots + (-1)^{n-1} \frac{1}{r_1^2}$$

$$= \frac{1}{r_{n+1}^2}.$$

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1 Math. Ann. **19**.

2 Crelle J. **114**.

Putting

$$\frac{1}{\rho_i} = +\sqrt{\frac{1}{r_i^2} - \frac{1}{r_{i-1}^2} + \ldots\ldots + (-1)^{i-1}\frac{1}{r_1^2}} = \frac{1}{R_i},$$

$$(i = 2, 3, \ldots\ldots, n-1)$$

$$\frac{1}{\rho_n} = \frac{1}{r_{n+1}} = -\sqrt{\frac{1}{r_n^2} - \frac{1}{r_{n-1}^2} + \ldots\ldots + (-1)^{n-1}\frac{1}{r_1^2}}$$

We have

$$\frac{da_i}{ds} = -\frac{{}_2a_i}{r_1},$$

$$\frac{d_2a_i}{ds} = \frac{{}_3a_i}{\rho_2} - \frac{{}_1a_i}{r_1}$$

(19)
$$\frac{d_3\mathfrak{n}_i}{ds} = \frac{{}_4a_i}{\rho_3} - \frac{{}_2a_i}{\rho_2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{d_na_i}{ds} = \frac{{}_{n+1}a_i}{\rho_n} - \frac{{}_{n-1}a_i}{\rho_{n-1}}$$

$$\frac{d_{n+1}a_i}{ds} = -\frac{{}_na_i}{\rho_n}$$

The formulas (18) and (19) are those which were given by Meyer[1] who defines K_r as the r^{th} curvature of the curve Γ with respect to the line g.

3 Dimensional Space.

8. Let $(\mathfrak{X})$ be a point orthogonal to (x) and on the tangent, $(\mathfrak{Y})$ orthogonal to (x) on the principal normal, $(\mathfrak{Z})$ orthogonal to (x) on the binormal, then

$$(x\mathfrak{X}) = (x\mathfrak{Y}) = (x\mathfrak{Z}) = (\mathfrak{X}\mathfrak{Y}) = (\mathfrak{X}\mathfrak{Z}) = (\mathfrak{Y}\mathfrak{Z}) = 0.$$

If we, hereafter, take as our parameter on the given curves the length of arc, then

[1] loc. cit.

$$x'_i=\frac{dx_i}{ds}, \quad (x'x')=1, \quad (i=0, 1, 2, 3)$$

and

$$\mathfrak{X}_i=kx'_i,$$

$$\mathfrak{Y}_i=-\frac{\rho}{k}(x_i+kx''_i),$$

$$\mathfrak{Z}_i=\rho\frac{\partial}{\partial t_i}\begin{vmatrix} t & x & x' & x'' \end{vmatrix},$$

$$(\mathfrak{X}\mathfrak{X})=k^2, \quad (\mathfrak{Y}\mathfrak{Y})=k^2, \quad (\mathfrak{Z}\mathfrak{Z})=k^2$$

and

$$\frac{d\mathfrak{X}_i}{ds}=\frac{\mathfrak{Y}_i}{\rho}-\frac{x_i}{k},$$

$$\frac{d\mathfrak{Y}_i}{ds}=-\frac{\mathfrak{X}_i}{\rho}-\frac{\mathfrak{Z}_i}{\tau},$$

$$\frac{d\mathfrak{Z}_i}{ds}=\frac{\mathfrak{Y}_i}{\tau}, \qquad (i=0, 1, 2, 3)$$

where ρ and τ are the curvature and torsion respectively.

9. If the curve Γ' be the evolute of the curve Γ or the curve traced by a point on the tangent of the curve P, then

$${}_0\eta_i=x_i,$$

$${}_1\eta_i=\mathfrak{X}_i,$$

$${}_2\eta_i=\mathfrak{Y}_i,$$

$${}_3\eta_i=-\mathfrak{Z}_i,$$

$$R_0=k, \quad R_1=\rho, \quad R_2=\tau, \quad \frac{1}{P_r}=0, \quad \frac{1}{Q_r}=0.$$

and we have

$$\frac{d_0\eta_i}{ds}=\frac{{}_1\eta_i}{R_0}, \quad \frac{d_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}-\frac{{}_0\eta_i}{R_0}$$

$$\frac{d_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}, \quad \frac{d_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2} \quad (i=0, 1, 2, 3)$$

which are the formulas given by Bianchi.[1]

10. Suppose I' be a curve traced by the point $(\mathfrak{y})$ which is orthogonal to (x) and on the principal normal. We get

$$a_i = \mathfrak{y}_i,$$

$${}_0\eta_i = x_i,$$

$${}_1\eta_i = \mathfrak{y}_i,$$

$${}_2\eta_i = k\frac{\mathfrak{y}'_i}{\sqrt{(\mathfrak{y}'\mathfrak{y}')}} = \frac{-\frac{\mathfrak{x}_i}{\rho} - \frac{\mathfrak{z}_i}{\tau}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}},$$

$${}_3\eta_i = \frac{\frac{\mathfrak{x}_i}{\tau} - \frac{\mathfrak{z}_i}{\rho}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}},$$

$$(i = 0, 1, 2, 3).$$

and

$$\frac{1}{R_1} = \sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

$$\frac{1}{R_2} = \frac{\tau\frac{d\rho}{ds} - \rho\frac{d\tau}{ds}}{\rho^2+\tau^2} = \frac{\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right) - \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)}{\frac{1}{\rho^2}+\frac{1}{\tau^2}}.$$

$$\frac{1}{P_0} = \frac{1}{k}, \quad \frac{1}{P_1} = 0, \quad \frac{1}{P_2} = 0,$$

$$\frac{1}{P_3} = \frac{1}{k^2\tau\rho}\,\frac{1}{\left(\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right) - \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right)},$$

$$\frac{1}{Q_0} = 0, \quad \frac{1}{Q_1} = 0, \quad \frac{1}{Q_2} = -\frac{1}{k\rho\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}}$$

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1 Lezioni di Geometria Differenziale, p. 457. A. Razzaboni, Le formale del Frenet in geometria iperbolica e laro applicazioni. (Bologna, Gamberini 1897). Also see Nischiuchi, loc. cit.

$$\frac{1}{Q_3} = -\frac{\sqrt{\frac{1}{\rho^2 k^2}\left(\frac{1}{\rho^2}+\frac{1}{\tau^2}\right)+E^2}}{k\tau E\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}},$$

$${}_0\xi_i = \mathfrak{X}_i, \quad {}_3\xi_i = -\frac{\frac{\mathfrak{X}_i}{\tau}-\frac{\mathfrak{Z}_i}{\rho}}{\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}} = -{}_3\eta_i,$$

$${}_2\zeta_i = x_i, \quad {}_3\zeta_i = \frac{1}{k}\,\frac{k\,E\,x_i+\frac{1}{\rho}\left(\frac{\mathfrak{Z}_i}{\rho}-\frac{\mathfrak{X}_i}{\tau}\right)}{\sqrt{\frac{1}{k^2\rho^2}\left(\frac{1}{\rho^2}+\frac{1}{\tau^2}\right)+E^2}}$$

$$(i=0, 1, 2, 3)$$

where

$$E=\frac{1}{\rho}\,\frac{d}{ds}\left(\frac{1}{\tau}\right)-\frac{1}{\tau}\,\frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Hence, we have

$$\frac{1}{R_1^2}\left(\frac{1}{Q_3^2}-\frac{1}{k^2 P_3^2}\right)=\frac{1}{k^2\tau^2},$$

$$\frac{1}{P_3 R_2}=\frac{1}{\rho\tau}\cdot\frac{1}{R_1^2},$$

$$\frac{1}{R_1 Q_2}+\frac{1}{k\rho}=0.$$

and

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0}, \quad \frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2},$$

$$\frac{d\,{}_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2}+\frac{{}_3\xi_2}{P_3}-\frac{{}_3\zeta_i}{Q_3}.$$

$$(i=0, 1, 2, 3)$$

11. If Γ and Γ' have the same principal normal, then the equations of the curve Γ' will be

$$\alpha_i = x_i \sin\frac{r}{k} + \mathfrak{y}_i \cos\frac{r}{k},$$

$$(i=0, 1, 2, 3),$$

and the coordinates of a points on the principal normal to Γ' and orthogonal to (α) will be

$$\overline{\mathfrak{y}}_i = x_i \cos\frac{r}{k} - \mathfrak{y}_i \sin\frac{r}{k},$$

$$(i=0, 1, 2, 3).$$

And we have

$$(\overline{\mathfrak{y}}\, d\alpha)=0$$

or

$$k^2\, dr=0$$

i. e.

$$r=\text{const.},$$

and we can find such quantities λ, μ, ν that

$$\overline{\mathfrak{y}}_i = \lambda\, \alpha_i + \mu\, \alpha'_i + \nu\, \alpha''_i .$$

Multiplying through α'_i, $\mathfrak{y}_i$ and adding, we get

$$\mu(\alpha'\, \alpha') + \nu(\alpha''\, \alpha') = 0,$$

$$\mu(\alpha'\, \mathfrak{y}) + \nu(\alpha''\, \mathfrak{y}) = 0.$$

Eliminating μ, ν,

$$\begin{vmatrix} (\alpha'\, \alpha') & (\alpha''\, \alpha') \\ (\alpha'\, \mathfrak{y}) & (\alpha''\, \mathfrak{y}) \end{vmatrix} = 0$$

or

$$k\left\{\frac{1}{\rho}\frac{d}{ds}\left(\frac{1}{\tau}\right) - \frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\rho}\right)\right\} = \tan\frac{r}{k}\,\frac{d}{ds}\left(\frac{1}{\tau}\right).$$

Hence, we have

$$\frac{k}{\tau}+\frac{Ck}{\rho}=C\tan\frac{r}{k} \quad (C\text{: integration const.})$$

This is the corresponding relation given by Bertrand in the Euclidean case.

And

$$\frac{1}{R_1}=\frac{\sqrt{(a'\,a')}}{k\cos\frac{r}{k}}=\frac{1}{\tau}\sqrt{\frac{1}{C^2}+1}\,,\quad \frac{1}{R_2}=0$$

$$\frac{1}{P_0}=\frac{1}{k}\,,\quad \frac{1}{P_1}=\frac{1}{k}\tan\frac{r}{k}\,,\quad \frac{1}{P_2}=0$$

$$\frac{1}{Q_o}=0,\quad \frac{1}{Q_1}=0,$$

$$\frac{1}{Q_2}=\frac{\sin\frac{r}{k}-\frac{k}{\rho}\cos\frac{r}{k}}{k\cos\frac{r}{k}\sqrt{(a'\,a')}}=\frac{1}{k\cos\frac{r}{k}\sqrt{1+C^2}}\,,$$

$${}_0\eta_i=x\,,\quad {}_1\eta_i=\mathfrak{y}_i,\quad {}_2\eta_i=\frac{k\,a'}{\sqrt{(a'\,a')}}\,,$$

$${}_0\xi_i=\mathfrak{x}_i,\quad {}_1\xi_i=-\mathfrak{x}_i,\quad {}_2\xi_i=\mathfrak{x}_i,$$

$${}_2\zeta_i=\cos\frac{r}{k}x_i-\sin\frac{r}{k}\mathfrak{y}_i=\overline{\mathfrak{y}}_i\,.$$

Hence, we have

$$\frac{1}{R_1}\frac{1}{Q_2}=\frac{1}{C\,k\cos\frac{r}{k}}\frac{1}{\tau}$$

$$=\frac{1}{k\cos\frac{r}{k}}\left[\frac{1}{k}\tan\frac{r}{k}-\frac{1}{\rho}\right]$$

or

$$\frac{1}{R_1}\frac{1}{Q_2}=\frac{1}{k\cos\frac{r}{k}}\left[\frac{1}{P_1}-\frac{1}{\rho}\right],$$

and the generalized Serret-Frenet formula will be

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2}, \qquad (i=0, 1, 2, 3)$$

And $({}_3\eta)$ will be on the absolute.

12. If the principal normal of the curve Γ be the binormal of the curve Γ', we have

$$a_i=x_i\sin\frac{r}{k}+{}_2y_i\cos\frac{r}{k}$$

$$(i=0, 1, 2, 3)$$

and let $(\overline{y})$ be the coordinates of the points on the binormal of Γ' and orthogonal to (a), then

$$\overline{y}_i=x_i\cos\frac{r}{k}-{}_2y_i\sin\frac{r}{k},$$

and we have

$$(a'\,\overline{y}_i)=0,$$

$$(a''\,\overline{y}_i)=0.$$

Hence we get

$$r=\text{const.},$$

$$\left(\frac{k^2}{\rho^2}+\frac{k^2}{\tau^2}-1\right)\sin\frac{r}{k}\cos\frac{r}{k}=\frac{k}{\rho}\left(\sin^2\frac{r}{k}-\cos^2\frac{r}{k}\right).$$

And

$$\frac{1}{R_1}=\frac{\sqrt{(a'\,a')}}{k\cos\frac{r}{k}},$$

$$\frac{1}{R_2}=\frac{k\cos\frac{r}{k}\sin\frac{r}{k}\left[\left(\sin\frac{r}{k}-\frac{r}{\rho}\cos\frac{r}{k}\right)\frac{d}{ds}\left(\frac{1}{\tau}\right)+\frac{k}{\tau}\frac{d}{ds}\left(\frac{1}{\tau}\right)\cos\frac{r}{k}\right]}{\sin\frac{r}{k}-\frac{k}{\rho}\cos\frac{r}{k}}$$

$$\frac{1}{P_0}=\frac{1}{k},\quad \frac{1}{P_1}=\frac{1}{k}\tan\frac{r}{k},\quad \frac{1}{P_2}=0,\quad \frac{1}{P_3}=0.$$

$$\frac{1}{Q_0}=0,\quad \frac{1}{Q_1}=0,\quad \frac{1}{Q_2}=\frac{1}{k}\tan\frac{r}{k}\sqrt{(a'\,a')}$$

$$\frac{1}{Q_3}=-\frac{1}{\tau\sqrt{(a'\,a')}}$$

Hence, we have

$$\frac{1}{Q_2}\,\frac{1}{Q_3}=-\frac{\tan\frac{r}{k}}{k\tau},$$

$$k^2\cos^2\frac{r}{k}\,\frac{d}{ds}\left(\frac{R_1}{Q_3}\right)+\frac{Q_2^2}{R_2}=0$$

and the generalized Serret-Frenet formula will be

$$\frac{d\,{}_0\eta_i}{ds}=\frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds}=\frac{{}_2\eta_i}{R_1}+\frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds}=\frac{{}_3\eta_i}{R_2}-\frac{{}_1\eta_i}{R_1}-\frac{{}_2\zeta_i}{Q_2}.$$

$$\frac{d\,{}_3\eta_i}{ds}=-\frac{{}_2\eta_i}{R_2}-\frac{{}_3\zeta_i}{Q_3}.$$

$$(i=0,\,1,\,2,\,3)$$

13. Suppose Γ' be a curve traced by the points (ξ) which is orthogonal to (x) and on the binormal, then

$$a_i=\xi_i$$

$$\frac{1}{R_1}=-\frac{1}{\tau},\quad \frac{1}{R_2}=\frac{1}{\rho},$$

$$\frac{1}{P_0}=\frac{1}{k},\quad \frac{1}{P_1}=0,\quad \frac{1}{P_2}=0,\quad \frac{1}{P_3}=0.$$

$$\frac{1}{Q_1}=0,\quad \frac{1}{Q_2}=0,\quad \frac{1}{Q_3}=\frac{1}{k}.$$

$$ {}_0\eta_i = x_i, \qquad {}_1\eta_i = \xi_i, \qquad {}_2\eta_i = \mathfrak{y}_i $$

$$ {}_3\eta_i = -\mathfrak{x}_i. \qquad {}_3\zeta_i = -x_i \qquad (i=0, 1, 2, 3) $$

And the generalized Serret-Frenet formula will be

$$ \frac{d\,{}_0\eta_i}{ds} = \frac{{}_0\xi_i}{P_0} $$

$$ \frac{d\,{}_1\eta_i}{ds} = \frac{{}_2\eta_i}{R_1}, \qquad \frac{d\,{}_2\eta_i}{ds} = \frac{{}_3\eta_i}{R_2} - \frac{{}_1\eta_i}{R_1} $$

$$ \frac{d\,{}_3\eta_i}{ds} = -\frac{{}_2\eta_i}{R_2} - \frac{{}_3\zeta_i}{Q_3} $$

$$ (i=0, 1, 2, 3). $$

14. If the binormal of the curve Γ be the principal normal to the curve Γ'' and let $(\mathfrak{y})$ be the coordinates of the point on the binormal of the curve Γ'' and orthogonal to (α), then

$$ \alpha_i = x_i \sin\frac{r}{k} + \xi_i \cos\frac{r}{k}, $$

$$ \overline{\mathfrak{y}}_i = x_i \cos\frac{r}{k} - \xi_i \sin\frac{r}{k} $$

$$ (i=0, 1, 2, 3) $$

and

$$ (\overline{\mathfrak{y}}\, d\alpha) = 0 $$

we get

$$ r = \text{const.} $$

More over we can find such quantities λ, μ, ν that

$$ \overline{\mathfrak{y}}_i = \lambda\, \alpha_i + \mu\, \alpha_i' + \nu\, \alpha_i'' $$

Multiplying through α_i', $\mathfrak{y}_i$ and adding, we have

$$ \mu(\alpha'\, \alpha') + \nu(\alpha''\, \alpha') = 0, $$

$$ \mu(\alpha'\, \mathfrak{y}) + \nu(\alpha''\, \mathfrak{y}) = 0 $$

or

$$\begin{vmatrix} (\alpha' {}'\alpha) & (\alpha' \alpha'') \\ (\alpha' \mathfrak{Y}) & (\alpha'' \mathfrak{Y}) \end{vmatrix} = 0$$

i. e.

$$\frac{1}{\rho}\tan^2\frac{r}{k} + k\frac{d}{ds}\left(\frac{1}{\tau}\right)\tan\frac{r}{k} + \frac{k^2}{\rho\tau^2} = 0.$$

and

$$\frac{1}{R_1} = \frac{\sqrt{(\alpha' \alpha')}}{k\cos\dfrac{r}{k}} = \sqrt{\frac{1}{k^2}\tan^2\frac{r}{k} + \frac{1}{\tau^2}}, \quad \frac{1}{R_2} = 0$$

$$\frac{1}{P_0} = \frac{1}{k}, \quad \frac{1}{P_1} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{P_2} = 0$$

$$\frac{1}{Q_0} = 0, \quad \frac{1}{Q_1} = 0, \quad \frac{1}{Q_2} = \frac{\tan\dfrac{r}{k}}{k\sqrt{(\alpha' \alpha')}}.$$

So we have

$$\frac{1}{R_1}\frac{1}{Q_2} = \text{const.} \quad \frac{1}{P_1} = \text{const.}$$

and

$$\frac{d\,{}_0\eta_i}{ds} = \frac{{}_0\xi_i}{P_0},$$

$$\frac{d\,{}_1\eta_i}{ds} = \frac{{}_2\eta_i}{R_1} + \frac{{}_1\xi_i}{P_1},$$

$$\frac{d\,{}_2\eta_i}{ds} = -\frac{{}_1\eta_i}{R_1} - \frac{{}_2\zeta_i}{Q_2}$$

where

$${}_0\eta_i = x_i, \quad {}_1\eta_i = \mathfrak{x}_i, \quad {}_2\eta_i = \frac{k\alpha'_i}{\sqrt{(\alpha' \alpha')}}$$

$${}_0\xi_i = \mathfrak{X}_i, \quad {}_1\xi_i = -\mathfrak{X}_i$$

$${}_2\zeta_i = \overline{\mathfrak{Y}}_i, \qquad (i = 0, 1, 2, 3).$$

And $({}_3\eta)$ will be on the absolute.

15. If the binormal of the curve Γ' be that of the curve Γ, and let $(\overline{\xi})$ be the coordinates of a points on the binormal to Γ' and orthogonal to (a), then

$$a_i = x_i \sin\frac{r}{k} + \xi_i \cos\frac{r}{k},$$

$$\xi_i = x_i \cos\frac{r}{k} - \xi_i \sin\frac{r}{k}$$

$$= \frac{\partial}{\partial t_i} \left| t\ a\ a'\ a'' \right|$$

$$(i = 0, 1, 2, 3),$$

and

$$(a'\ \overline{\xi}) = (a''\ \overline{\xi}) = 0.$$

Hence, we have

$$r = \text{const.}$$

$$\frac{1}{\tau^2} = \frac{1}{k^2}.$$

In this case, it is evident that the binormals will generate a ruled surface with vanisching Gaussian curvature.[1]

And

$$\frac{1}{R_1} = \frac{1}{k\cos\frac{r}{k}}, \quad \frac{1}{R_2} = \frac{1}{\rho}.$$

$$\frac{1}{P_0} = \frac{1}{k}, \quad \frac{1}{P_1} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{P_2} = 0, \quad \frac{1}{P_3} = 0.$$

$$\frac{1}{Q_0} = 0, \quad \frac{1}{Q_1} = 0, \quad \frac{1}{Q_2} = \frac{1}{k}\tan\frac{r}{k}, \quad \frac{1}{Q} = \frac{1}{k}$$

$${}_3\eta_i = {}_2\mathfrak{y}_i \sin\frac{r}{k} - \mathfrak{X}_i \cos\frac{r}{k},$$

$${}_2\eta_i = \mathfrak{X}_i \sin\frac{r}{k} + {}_2\mathfrak{y}_i \cos\frac{r}{k},$$

---

1 Coolidge. Non-Euclidean Geometry.

$$
\begin{aligned}
{}_1\eta_i &= \mathfrak{Y}_i, \\
{}_1\xi_1 &= -\mathfrak{X}_i, \\
{}_3\xi_i &= \mathfrak{Y}_i \sin\frac{r}{k} - x_i \cos\frac{r}{k} \\
{}_2\zeta_i &= x_i \cos\frac{r}{k} - \mathfrak{Y}_i \sin\frac{r}{k}
\end{aligned}
$$

$$(i=0, 1, 2, 3)$$

So the generalized formula will be

$$
\begin{aligned}
\frac{d\,{}_0\eta_i}{ds} &= \frac{{}_0\xi_i}{P_0}, \\
\frac{d\,{}_2\eta_i}{ds} &= \frac{{}_2\eta_i}{R_1} + \frac{{}_1\xi_i}{P_1} \\
\frac{d\,{}_2\eta_i}{ds} &= \frac{{}_3\zeta_i}{R_2} - \frac{{}_1\eta_i}{R_1} - \frac{{}_2\zeta_i}{Q_2} \\
\frac{d\,{}_3\eta_i}{ds} &= -\frac{{}_2\eta_i}{R_2} - \frac{{}_3\zeta_i}{Q_3}
\end{aligned}
$$

$$(i=0, 1, 2, 3)$$

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CONTENTS.

On the Principle of the Light-Velocity.
By Toshizô Matsumoto 187

Projective Differential Geometry of Nondevelopable Surfaces I.
By Jôyô Kanitani 191

On the Curvature of a Curve in *n* Dimensional Non-Euclidean Space.
By Hidetoshi Kashiwagi. 221